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Short Communication

Geometrical criteria on the higher order smoothness of composite surfaces

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Abstract

A generalization of a theorem by Pegna and Wolter—called Linkage Curve Theorem—is presented. The new theorem provides a condition for joining two surfaces with high order geometric continuity of arbitrary degree n. It will be shown that the Linkage Curve Theorem can be generalized even for the case when the common boundary curve is only G^1.

Keywords: Linkage curve theorem; Higher order smoothness

1. Introduction

The construction of smooth, composite surfaces by joining adjacent surfaces is still an interesting research topic in CAGD. Joining surfaces may occur along either a constant parameter line or an arbitrary surface curve shared by the two surfaces. A typical example for the former situation is the construction of a composite surface by merging parametric patches along their borders. For the latter one the most important case to be considered is blending, where a smooth transition surface needs to be joined to a surface with high order smoothness along a contact curve.

Two theorems on joining curvature continuous surfaces were proved in (Pegna and Wolter, 1992). The first one, called the *Three Tangents Theorem* states the following:

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Theorem A. *Two surfaces tangent at a point* $\mathbf{p}_0$ *have the same normal curvatures if and only if their normal curvatures agree in three tangent directions, of which any pair is linearly independent.*

The second theorem was called *Linkage Curve Theorem.* (A curve is called linkage curve if it is common surface curve of both surfaces.)

Theorem B. *Two surfaces tangent along a* C^1*-smooth linkage curve are curvature continuous if and only if at every point of the linkage curve, their normal curvature agrees for a direction other than the tangent to the linkage curve.*

The generalization of the Three Tangents Theorem for higher order of smoothness was addressed in (Wolter and Tuohy, 1992, p. 256) cf. Corollary 1 "$n+1$" Tangents Theorem, (characterization of $n+1$ order surface contact at a point). In this paper a generalization of the Linkage Curve Theorem is given. The outline of the paper is the following. In Section 2 the concept of "higher order smoothness" will be briefly introduced. In Section 3 the generalization of Theorem B is given. Finally we summarize our results in Section 4.

In the paper bold letters will denote vectors from $\mathbb{R}^3$.

2. Higher order smoothness of curves and surfaces

We shall call a curve (surface) G^n continuous if there is a representation of it with a regular C^n map from a closed, bounded interval (or from a compact, simply connected domain [3] in $\mathbb{R}^2$) into $\mathbb{R}^3$. As usual regularity means here that the first order differentials of the curve (or surface) are of full rank. Note that this property is preserved at every regular C^n *reparameterization* of the curve (or surface).

(See further details in (DeRose and Barsky, 1985; Gregory, 1989; Herron, 1987).)

We say that two surfaces have a G^n join if they are G^n continuous, their intersection contains a curve and if we consider their restrictions to one side of this curve the union of these parts form a G^n surface.

We shall need the following simple lemma:

Lemma 1. *Let* $\mathbf{F}:\mathbb{R}^2\to\mathbb{R}^3$ *be a* G^n *surface, where* $n>0$ *and let us take an arbitrary, but fixed point* $\mathbf{p}$ *on* $\mathbf{F}$. *Consider a coordinate system transformation where the origin moves to* $\mathbf{p}$ *and the direction of the* z*-axis points towards the surface normal at* $\mathbf{p}$. *The direction of the* x*-axis in the tangent plane is arbitrary, but fixed as well (and so the* y*-axis is also determined). Then there is an open* $V\subset\mathbb{R}^3$ *and an open* $U\subset\mathbb{R}^2$ *and an* n*-times continuously differentiable* $f(x,y)$ *function such that* $\mathbf{p}\in\mathbf{F}\cap V$ *and* $(x,y,f(x,y))=\mathbf{F}\subset V$ *when* $(x,y)\in U$.

Proof. This follows directly from the definition by the Implicit Function Theorem, see also (Pegna and Wolter, 1992). □

[3] We could consider more general domains in $\mathbb{R}^2$ but this is irrelevant for this paper.

Remark. Sometimes this $f(x, y)$ representation is called the Euler–Monge form of the surface **F**.

Obviously if $f(x, y) \in C^n$ then $(x, y, f(x, y))$ is a G^n surface.

Similar statements are true for curves.

The lemma will be used in the proofs of the forthcoming Theorem 1. The significance of the lemma is that it gives a common parametrization for all surfaces which are incident to a given point **p** and have a common tangent plane there in a neighbourhood of **p**.

3. Linkage Curve Theorem for C^n surfaces

Now we give a proper generalization of the Linkage Curve Theorem for G^n surfaces.

Theorem 1. *Let* **F** *and* **G** *be* G^n *surfaces sharing a common* G^1 *curve denoted by* $\mathbf{R}(t)$. *Suppose that there exists a family of* G^n *curves* $\mathbf{E}_t(s) = \mathbf{E}(t, s)$ *so that each* $\mathbf{E}_t$ *is a surface curve of* **F** *for* $s \leqslant 0$, *each* $\mathbf{E}_t$ *is a surface curve of* **G** *for* $s \geqslant 0$, *and* $\mathbf{E}_t(0) = \mathbf{R}(t)$ *and* $\mathbf{E}_t'(0)$ *is not parallel to* $\mathbf{R}'(t)$. *Then* **F** *and* **G** *have a* G^n *continuous join.*

Remark. A similar statement was proved in (Gregory, 1989). The main difference is that here only G^1 continuity is required for **R**. This is not important if one wants to apply the theorem for joining patches along parameter lines, but it can be important when the common surface curve is not a parameter line, for example in the case of blending.

Proof of Theorem 1. We prove the theorem by induction for n.

If $n = 1$ then we have to prove that **F** and **G** have a G^1 join. This is trivial since in every point of $\mathbf{R}(t)$ the normal vectors of both **F** and **G** are parallel to the cross-product

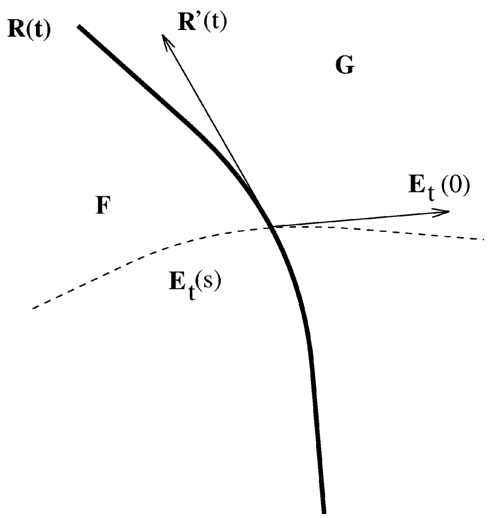

Fig. 1. Join of surfaces **F** and **G**.

of $\mathbf{E}_t'(0)$ and $\mathbf{R}'(t)$ which is not $\mathbf{0}$ due to our conditions. Hence both surfaces **F** and **G** share a common tangent plane along the G^1 linkage curve $\mathbf{R}(t)$. According to (Pegna and Wolter, 1992, p. 208), this implies the existence of a C^1 continuous Euler–Monge form representing locally the union of the surfaces **F** and **G**.

Now suppose that the theorem holds for $n-1$ and we want to prove it for n. Notice that from the induction condition it follows that **F** and **G** have a G^{n-1} join.

Let $\mathbf{p}_0 = \mathbf{R}(t_0)$ be an arbitrary, but fixed point. It is enough to prove the G^n join in the neighborhood of this point. Now fix our coordinate system so that $\mathbf{p}_0$ is the origin and the z-axis is parallel to the surface normal there. The x and y-axes are orthogonal to each other and the z-axis. There is a neighborhood of $\mathbf{p}_0$ so that **F** and **G** can be represented as $(x, y, f(x, y))$ and $(x, y, g(x, y))$ using suitable C^n functions. Let

$$\mathbf{R}(t) = \bigl(\alpha(t), \beta(t), h\bigl(\alpha(t), \beta(t)\bigr)\bigr)$$

and

$$\mathbf{E}_t(s) = \bigl(\varphi_t(s), \psi_t(s), h\bigl(\varphi_t(s), \psi_t(s)\bigr)\bigr),$$

where h is equal to f or g depending on the sign of s (or in the case of **R** it can be either of them). Here α and β are C^1, $\varphi_t(s)$ and $\psi_t(s)$ are C^n functions.

From the induction condition it follows that all partial derivatives of f and g are equal up to the $(n-1)$th order:

$$\frac{\partial^m f(\alpha(t), \beta(t))}{\partial x^k \partial y^{m-k}} = \frac{\partial^m g(\alpha(t), \beta(t))}{\partial x^k \partial y^{m-k}} \quad (k = 0, \ldots, m;\ m < n). \tag{1}$$

If we can prove that the nth order partial derivatives are equal too then the proof is complete. Now let us use the condition that $\mathbf{E}_t(s) \in C^n$. Having differentiated $\mathbf{E}_t(s)$ n-times with respect to the variable s we obtain:

$$\sum_{j=0}^{n} \frac{\partial^n f}{\partial x^j \partial y^{n-j}} \binom{n}{j} \dot{\varphi}^j \dot{\psi}^{n-j} + \text{terms with lower order}$$
$$- \sum_{j=0}^{n} \frac{\partial^n g}{\partial x^j \partial y^{n-j}} \binom{n}{j} \dot{\varphi}^j \dot{\psi}^{n-j} - \text{terms with lower order} = 0. \tag{2}$$

As we have already remarked, the lower order terms are equal and so they cancel out from the equation.

For $m = n-1$, both sides of (1) are continuously differentiable functions of t. After differentiating with respect to the variable t we have:

$$\frac{\partial^n f}{\partial x^{k+1} \partial y^{n-k-1}} \dot{\alpha} + \frac{\partial^n f}{\partial x^k \partial y^{n-k}} \dot{\beta} - \frac{\partial^n g}{\partial x^{k+1} \partial y^{n-k-1}} \dot{\alpha} - \frac{\partial^n g}{\partial x^k \partial y^{n-k}} \dot{\beta} = 0$$
$$(k = 0, \ldots, n-1). \tag{3}$$

Now we have $n+1$ equations for the nth order partial derivatives by (2) and (3). More precisely, let the unknowns be the *differences* of the corresponding partial derivatives then we have a system with $n-1$ equations. The nth equation is (2). The right side is 0, so if the matrix of the system is non-singular then each unknown difference is zero, i.e., the partial derivatives $(k = 0, \ldots, n)$ are also equal to each other and we proved our assertion.

From (3) and (2) the determinant is the following:

$$D = \begin{vmatrix} \dot\beta & \dot\alpha & & & & \\ & \dot\beta & \dot\alpha & & & \\ & & \dot\beta & \dot\alpha & & \\ & & & & \dots & \\ b_0 & b_1 & b_2 & b_3 & \dots & b_n \end{vmatrix},$$

where $b_j = \binom{n}{j}\dot\varphi^j\dot\psi^{n-j}$. After some algebra one obtains:

$$D = \sum_{j=0}^{n} \binom{n}{j} \dot\varphi^j \dot\psi^{n-j} (-1)^j \dot\beta^j \dot\alpha^{n-j} = (\dot\alpha\dot\varphi - \dot\beta\dot\psi)^n.$$

Therefore the determinant is equal to 0 if and only if $\mathbf{E}_t(s)$ is tangential to $\mathbf{R}(t)$, but this was excluded by the condition of Theorem 1. □

4. Conclusion

In this paper a generalization of a theorem by Pegna and Wolter was described to extend their idea for higher order smoothness between two adjacent surfaces.

It is obvious that the conclusion of Theorem 1 remains valid if we require the linkage curve to be piecewise differentiable only. It is an interesting problem whether any kind of further weakening of the related conditions is possible?

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