Geometrical Criteria to Guarantee Curvature Continuity of Blend Surfaces

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Computer Aided Geometric Design (CAGD) of surfaces sometimes presents problems that were not envisioned in classical differential geometry. This paper presents mathematical results that pertain to the design of curvature continuous blending surfaces. Curvature continuity across normal continuous surface patches requires that normal curvatures agree along all tangent directions at all points of the common boundary of two patches, called the linkage curve. The Linkage Curve theorem proved here shows that, for the blend to be curvature continuous when it is already normal continuous, it is sufficient that normal curvatures agree in one direction other than the tangent to a first order continuous linkage curve. This result is significant for it substantiates earlier works in computer aided geometric design. It also offers simple practical means of generating second order blends for it reduces the dimensionality of the problem to that of curve fairing, and is well adapted to a formulation of the blend surface using sweeps. From a theoretical viewpoint, it is remarkable that one can generate second order smooth blends with the assumption that the linkage curve is only first order smooth. The geometric criteria presented may be helpful to the designer since curvature continuity is a technical requirement in hull or cam design problems. The usefulness of the linkage curve theorem is illustrated with a second order blending problem whose implementation will not be detailed here.

1 Introduction

Computer Aided Geometric Design of surfaces sometimes presents problems that were not envisioned in classical differential geometry. One such problem is in the area of blending and fairing from which this paper is derived.

A blend is a surface patch connecting smoothly two or more other prescribed smooth surfaces. Blends occur often in engineering. They may appear as by-products of the manufacturing process because sharp angles cannot be generated; for example milling of concave corners with a ball cutter. Blends appear as a design requirement for safety or aesthetic reasons; for example, fairing of a car body. Finally, blends may also turn out to be a functional design requirement. In cam design, for example, a blend must connect two prescribed surfaces without second order discontinuity, for such discontinuities would create abrupt changes in acceleration. Aircraft, ship, and submarine design are other examples in which groups of functional surfaces are required to be smoothly connected without second order geometric discontinuities. Examples of such functional groups are wing and body, keel and hull, or bow and hull, that must be second order smooth to reduce the risk of flow separation and turbulence.

Because differential geometry is concerned with analysis and not design of surfaces, mathematics textbooks do not address the problem of joining surface patches with a prescribed order of continuity; instead, those textbooks usually assume that continuity properties at the joint between two patches are satisfied up to a prescribed order. The engineer, nonetheless, is faced with the problem of constructing blending surfaces so that they satisfy a set of prescribed technological constraints, that includes continuity at the joint. Even though the mathematician and the engineer use the same methods, some theoretical problems may arise that are specific to the design process, we shall refer to this class of problems as "*engineering geometry.*"

This paper presents some theoretical results that pertain to engineering geometry. The main contribution herein is a theorem, stated in Section 4, that identifies a simple practical geometric criterion to guarantee second order smoothness of composite surface patches. This theorem is an important theoretical foundation for related works in second order continuous blending of functional surfaces by Pegna [1]; it also substantiates or clarifies earlier claims in the same area of geometric modeling by Hansmann [2] and Kahmann [3].

The paper is organized in four parts. Section 2 reviews the basic principles and formulates the problem at hand. Section 3 reviews earlier related works. Sections 4 and 5 are the core and main contribution of this paper; they contain two practical theorems identifying sufficient conditions for second order smoothness. Section 6 discusses and illustrates the usefulness of the criteria identified in Sections 4 and 5.

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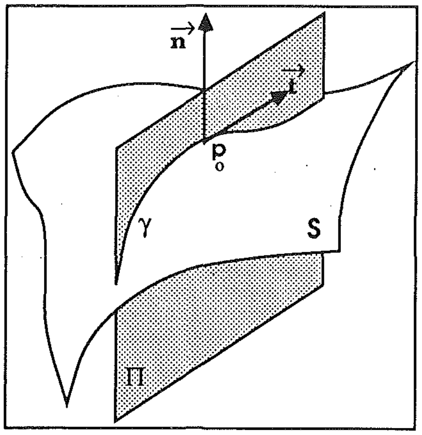


Fig. 1 A surface patch S and its normal section by a plane Π at point p_0 define the tangent direction t and the normal curvature of the surface at point p_0.

2 Background

This section reviews briefly the notions of normal curvature of a surface and curvature continuity for surface patches. The reader experienced with differential geometry may wish to omit this part and go directly to section 3.

2.1 Normal Curvature of a Surface. Define a surface S, say as graph of a twice differentiable function f over the x,y plane (Fig. 1). Let $\mathbf{n}$ be the unit surface normal to S at point $\mathbf{p}_0$ and Π a normal plane section through $\mathbf{n}$. The intersection curve γ has a unit tangent $\mathbf{t}$ at $\mathbf{p}_0$. The arc length is measured on the curve γ is obtained as the integral of the *first fundamental form* [4]

$$ds^2 = E\ dx^2 + 2F\ dx\ dy + G\ dy^2 \tag{1}$$

with

$$E = 1 + f_{,x}^2 \qquad F = f_{,x} f_{,y} \qquad G = 1 + f_{,y}^2 \tag{2}$$

The first fundamental form can also be written in a matrix form as:

$$ds^2 = (dx,\ dy) \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}. \tag{3}$$

The symmetric matrix $g = [\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}]$ is said to be the components of the *first fundamental* or *metric tensor*.

The curvature κ_0 of the intersection curve γ at $\mathbf{p}_0$ is called the normal curvature of the surface S at $\mathbf{p}_0$ in the direction of $\mathbf{t}$. Let λ be the ratio dy/dx in the direction of the tangent.

$$\lambda = \frac{\dfrac{dy}{ds}}{\dfrac{dx}{ds}} = \frac{dy}{dx}. \tag{4}$$

Then the normal curvature κ_0 is obtained as the ratio of the second to the first fundamental form [4],

$$\kappa_0 = \frac{L + 2M\ \lambda + N\ \lambda^2}{E + 2F\ \lambda + G\ \lambda^2} \tag{5}$$

with

$$L = \frac{f_{,xx}}{\sqrt{1 + f_{,x}^2 + f_{,y}^2}} \qquad M = \frac{f_{,xy}}{\sqrt{1 + f_{,x}^2 + f_{,y}^2}} \qquad N = \frac{f_{,yy}}{\sqrt{1 + f_{,x}^2 + f_{,y}^2}}. \tag{5}$$

Note that equation (5) can be written equivalently as follows, for a tangent direction (α, β) on the surface, i.e. this tangent direction corresponds to the tangent vector

$$\kappa_0 = \frac{L\ \alpha^2 + 2M\ \alpha\beta + N\ \beta^2}{E\ \alpha^2 + 2F\ \alpha\beta + G\ \beta^2}. \tag{6}$$

Like the first fundamental form, the second fundamental form can also be written in matrix format,

$$L\ \alpha^2 + 2M\ \alpha\beta + N\ \beta^2 = (\alpha, \beta) \begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{7}$$

The symmetric matrix $b = [\begin{smallmatrix} L & M \\ M & N \end{smallmatrix}]$ is said to be the components of the *second fundamental tensor*. Since b is symmetric, its eigenvalues are real and their associated eigenvectors are orthogonal. Note that the elements of b are the components of the curvature tensor in natural coordinates. Thus the eigenvalue problem is expressed as $\det(b - \kappa\ g) = 0$, or equivalently $\det(g^{-1}\ b - \kappa\ I) = 0$, where κ denotes the eigenvalues. The eigenvalues κ_1 and κ_2 of b are called *principal curvatures* and their inverse $\rho_1 = 1/\kappa_1$ and $\rho_2 = 1/\kappa_2$ *principal radii of curvature*. The corresponding eigenvectors define the principal directions of curvature on the surface **S** at point $\mathbf{p}_0$. Curves on the surface that are always tangent to a principal direction of curvature are called lines of principal curvature. They form a network of orthogonal curves on the surface. The determinant of b is equal to $\kappa_1 \kappa_2 \det(g)$. The product $\kappa_1 \kappa_2$ is called the *Gaussian curvature*.

2.2 Dupin's Indicatrix. Let ξ and η be unit vectors along the principal directions of curvature. The normal curvature for a tangent unit vector $\mathbf{t}$ with coordinates (α, β) in the basis $\{\xi, \eta\}$ takes on a very simple form, known as Euler's theorem [4].

Nomenclature

- $\in$ = set membership
- $\subset$ = set inclusion
- $\cap$ = set intersection
- o = null set (or empty set)
- $\mathcal{C}(\mathfrak{I})$ = image of a set $\mathfrak{I}$ created by the map $\mathcal{C}$
- $\mathcal{R}$ = set of real numbers
- $\mathbf{p}$ = point
- $\mathbf{n}$ = vector
- $\|\mathbf{n}\|$ = norm of the vector
- $\langle \mathbf{u}, \mathbf{v} \rangle$ = inner product of two vectors
- $\wedge$ = cross-product of two vectors
- span($\mathbf{u}, \mathbf{v}$) = space spanned by the two vectors $\mathbf{u}$, $\mathbf{v}$
- $|\rho|$ = absolute value of the scalar ρ
- Det(L) or $|L|$ = determinant of the matrix L
- $f_{,x}$ or $\partial_x f$ = partial derivatives of f with respect to x
- $f_{,xy}$ = double partial derivative
- C^2 = second order continuity
- D_f = Jacobian of f
- $\circ$ = composition of functions
- ■ = end of proof

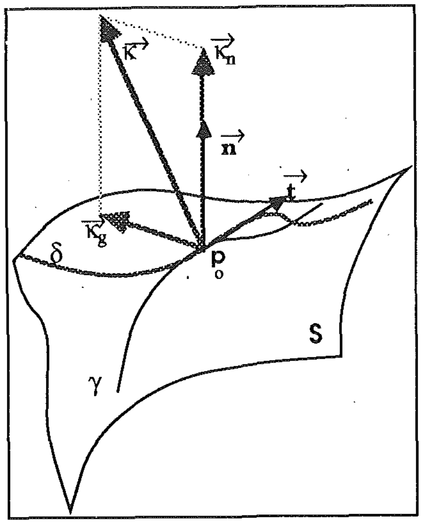


Fig. 2 A second order continuous curve δ drawn on the surface S through point p_0 has two curvature components $\vec{\kappa}_g$ and $\vec{\kappa}_n$ respectively tangent and normal to S at p_0 called the geodesic and the normal curvature.

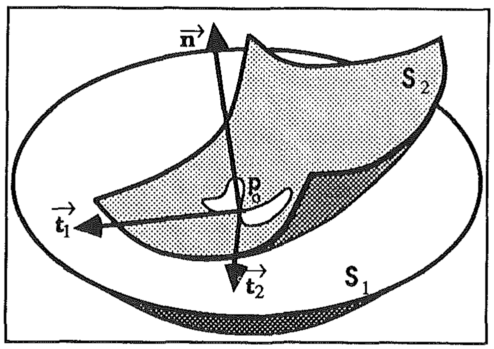


Fig. 3 Two surface patches S_1 and S_2 are tangent at point p_0. They intersect and have the same normal curvature along directions t_1 and t_2 at p_0.

$$\kappa = \kappa_1 \alpha^2 + \kappa_2 \beta^2 \tag{8}$$

Let ψ be the polar angle $(\vec{t}, \vec{\xi})$, and let ρ be the normal curvature radius. Recall that $\vec{t}$ is a unit vector ($\alpha^2 + \beta^2 = 1$, $\cos\psi = \alpha$, $\sin\psi = \beta$). One can rewrite equation (8) to plot $\sqrt{|\rho|}$ in polar coordinates.

$$\frac{\rho \cos^2\psi}{\rho_1} + \frac{\rho \sin^2\psi}{\rho_2} = 1 \tag{9}$$

Equation (9) is the equation of a conic, known as *Dupin's indicatrix* [4]. If the Gaussian curvature is positive, then Dupin's indicatrix is an ellipse. If one of the principal curvatures is null (i.e., the Gaussian curvature is null) then Dupin's indicatrix is a set of two lines parallel to the axis with non-null curvature. If the Gaussian curvature is negative, Dupin's indicatrix is a hyperbola; note however that one pair of opposite branches of the hyperbola corresponds to negative normal curvature and the other pair to positive normal curvature. This fact is often understated in mathematics textbooks or omitted in the CAGD literature. We shall see however that this remark is relevant to properties proved in Section 4.

2.3 Normal and Geodesic Curvatures on a Surface. A second order continuous curve δ drawn on the surface S through a point $\mathbf{p}_0$ has a curvature vector κ with one component κ_g tangent to S at $\mathbf{p}_0$ and a component normal $\kappa_n = \kappa_n\ \mathbf{n}$. The tangent component κ_g is called the *geodesic curvature*. The normal component κ_n is called the *normal curvature* (Fig. 2). The geodesic curvature is an original property of the curve δ

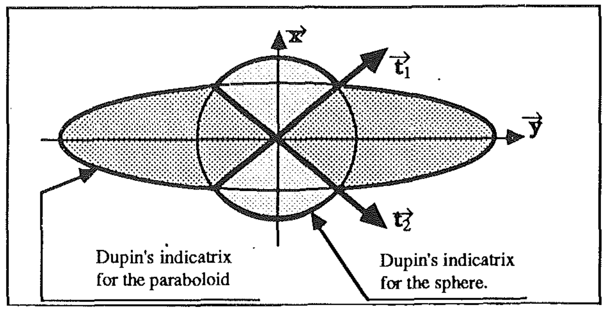


Fig. 4 The Dupin's indicatrices (as seen from the negative z direction) show that the paraboloid and the sphere intersect and have the same normal curvatures along two directions $\vec{t}_1$ and $\vec{t}_2$.

drawn on the surface S, as it is determined by the curvature of the curve δ in the tangent plane of S. The normal curvature, however, is not a property of the curve δ; it is imposed by the surface S and is the same for all C^2 curves with tangent **t** at $\mathbf{p}_0$. The normal curvature of a curve with tangent **t** at $\mathbf{p}_0$ is given by equation (6).

Hence the normal curvature $\kappa_n(\mathbf{p}_0, \mathbf{t})$ in any direction **t** is an original geometric property of the surface and is completely determined by the knowledge of the metric and curvature tensors at point $\mathbf{p}_0$. The curvature tensor characterizes the second order differential properties of a surface.

2.4 Osculating Surfaces. Consider two surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$ defined as graphs of two twice differentiable functions h and m over the x,y plane. Those surfaces $\mathbf{S}_1$, $\mathbf{S}_2$ can be tangent (to each other) at one point $\mathbf{p}_0 = (x_0, y_0, z_0)$. The normal curvatures of $\mathbf{S}_1$ and $\mathbf{S}_2$ may agree at $\mathbf{p}_0$ for two tangent directions $\mathbf{t}_1$ and $\mathbf{t}_2$. But the normal curvatures need not agree at $\mathbf{p}_0$ for all tangent directions of $\mathbf{S}_1$, $\mathbf{S}_2$ at $\mathbf{p}_0$ (Fig. 3). Hence the second derivatives of h and m need not agree at (x_0, y_0). (The first derivatives of h and m agree at (x_0, y_0), because $\mathbf{S}_1$, $\mathbf{S}_2$ are tangent at (x_0, y_0, z_0), see Assertion 1, Section 5).

Consider for example the elliptic paraboloid with equation $z = x^2 + y^2/4$. The principal curvature directions at (0, 0) are the x and y axes. The respective principal curvatures are 2 and 1/2. A sphere of radius ρ centered at $(0,0,\rho)$ is tangent to the paraboloid at (0,0,0). When $1/2 < \rho < 2$, the sphere intersects the paraboloid at (0,0) along two directions that are symmetric with respect to the x and y axes. This can be seen clearly by mean of the Dupin's indicatrices of the paraboloid and the sphere at (0,0) (Fig. 4).

The Dupin indicatrices of the paraboloid and the sphere are respectively the ellipse of equation $2x^2 + y^2/2 = 1$ and the circle of equation $x^2 + y^2 = \rho$. The plot on Fig. 4 shows that the spherical cap and the paraboloid intersect and have the same normal curvatures along two tangent directions $\mathbf{t}_1$ and $\mathbf{t}_2$ that are symmetric with respect to the x and y axes. The dark and light gray shaded areas in Fig. 4 correspond to the visible sectors of the paraboloid and the sphere respectively. Note that the normal curvatures of the paraboloid and the sphere may agree at (0,0) for two tangent (even orthogonal) directions $\mathbf{t}_1$ and $\mathbf{t}_2$ and yet, they do not agree for all tangent directions. Figure 5(*a*) illustrates the same claim for pairs of surfaces with positive and negative Gaussian curvatures. Figure 5(*b*) shows that two normal continuous surfaces may share a common set of Dupin indicatrices along their linkage curve and yet not be curvature continuous.

2.5 Curvature Continuity of Composite Surface Patches in Engineering Design. Consider two surface patches $\mathbf{S}_1$ and $\mathbf{S}_2$ defined as graphs of two twice differentiable functions h and m over domains of the x,y plane. Assume that $\mathbf{S}_1$ and $\mathbf{S}_2$ share a common boundary curve λ. Using Hansmann's terminology [2], the curve λ is called a *linkage curve*.

For first order smoothness of the composite surface, the

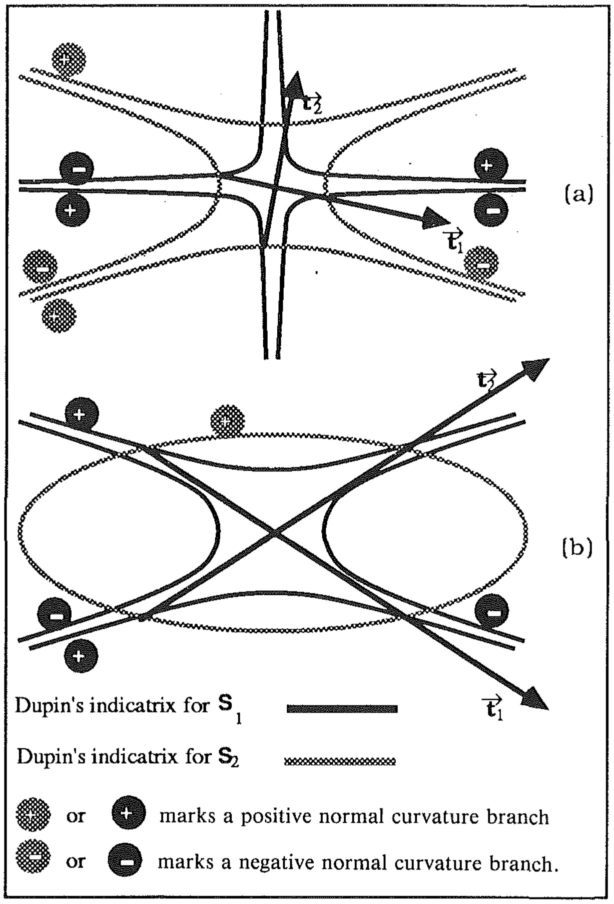


Fig. 5(*a*) As can be seen graphically for various pairs of Dupin's indicatrices, the normal curvatures may agree for two tangent directions, but need not agree for all tangent directions. Note that only intersecting branches corresponding to a same curvature sign actually yield a common normal curvature.

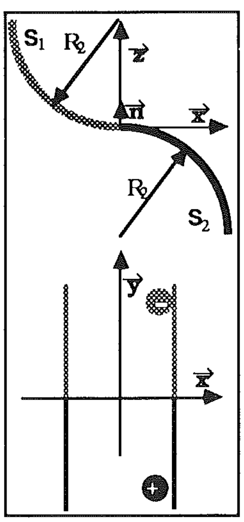


Fig. 5(*b*) End view of two cylinders with same radius, parallel axes and connecting along a common generatrix (top) and their common Dupin's indicatrices (bottom). This example shows that two surfaces may have the same Dupin indicatrices along a common linkage curve and yet not be curvature continuous because the signs of the normal curvatures do not agree.

tangent planes to S_1 and S_2 must agree at all points of the linkage curve. This is referred to as the *first order blending problem*. First order smoothness is a common requirement in engineering for safety (no sharp edges) and for aesthetic reasons. However first order smoothness is not sufficient for many mechanical design constraints. Consider for example a cam surface made of cylinder patches (Fig. 6). Even though the principal directions of curvatures agree at the linkage curve, the radii of curvature do not. A mass particle moving on the surface across the linkage curve will experience an acceleration discontinuity, which is not desirable for most practical purposes. For more general examples consider Fig. 7.

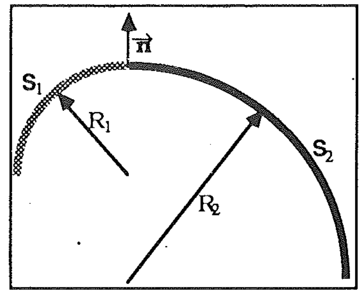


Fig. 6 A cam profile made of cylinder patches. Even though the cylindrical patches are normal continuous, their normal curvatures do not agree. A particle moving along the surface will experience an acceleration discontinuity at the linkage curve.

In the three examples of Fig. 7, a surface is made of two patches S_1 (clear) and S_2 (dark) connecting with various degrees of geometric continuity. The patch S_1 is kept constant throughout the examples. The boundary conditions on patch S_2 are adjusted to meet geometric continuity in the top case, surface unit normal continuity in the middle case, and curvature continuity in the bottom case. The lines of principal curvature experience a tangent discontinuity across the linkage curve in the top and middle cases. Respectively in the top, middle, and bottom cases, the lines of curvature experience a tangent discontinuity in 3-space, a tangent discontinuity confined to the local tangent plane, and no tangent discontinuity. Note that in the top two cases, a curve lying on the surface will experience at least a curvature discontinuity when crossing the linkage curve; for example geodesics (locally shortest paths on the surface) will experience a curvature discontinuity when crossing the linkage curve joining the two cylindrical patches of Fig. 6. The flow of a Newtonian fluid (water for example) over the surface will experience an acceleration discontinuity that may result in flow separation and turbulence. Note finally that when the surface is curvature continuous (bottom case) the lines of curvature are tangent continuous. The converse of this statement is not true. For example in Figs. 5(*b*) and 6, the lines of curvatures are tangent continuous and yet the surface is not curvature continuous across the linkage curve.

2.6 Characterization of Curvature Continuity. This section presents a brief survey of curvature continuity criteria commonly encountered in the literature. Curvature continuity across the linkage curve on a second order smooth surface is obtained when normal curvatures agree in every directions at every point on the linkage curve. This requirement can be equivalently formulated in other ways:

- **The osculating paraboloids of the two patches agree at all points of the linkage curve.**

If we assume that the surface patches are normal continuous across the linkage curve, then the following criteria apply:

- **The Dupin's indicatrices and normal curvature signs of the two patches agree at all points of the linkage curve.** Note that the requirement that the curvature signs have to agree is often omitted in the literature. As shown by the example in Fig. 5 however, Dupin's indicatrices may agree while the normal curvatures do not.
- **The asymptotic directions of the two patches agree at all points of the linkage curve** [3]. As will be shown in Section 5,

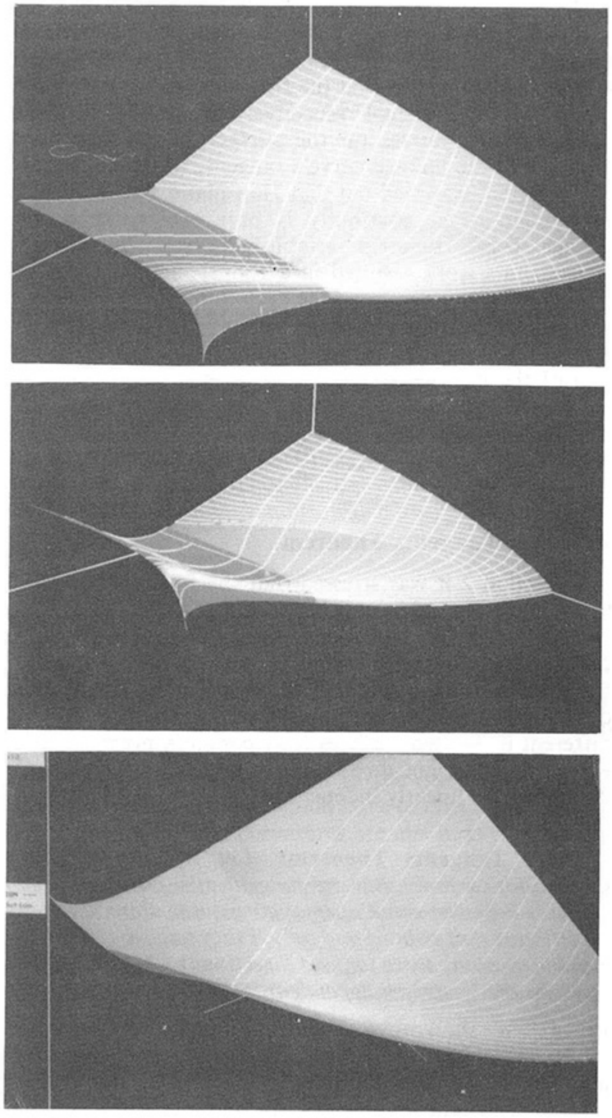


Fig. 7 Behavior of the lines of curvature across the linkage curve. For all three examples, the clear patch is constant. Local boundary conditions are chosen for the dark patch to ensure continuity (top), normal continuity (middle), and curvature continuity (bottom). In the top, middle, and bottom pictures, the lines of curvature experience respectively a tangent discontinuity in space, a tangent discontinuity confined to the local tangent plane, and no tangent discontinuity.

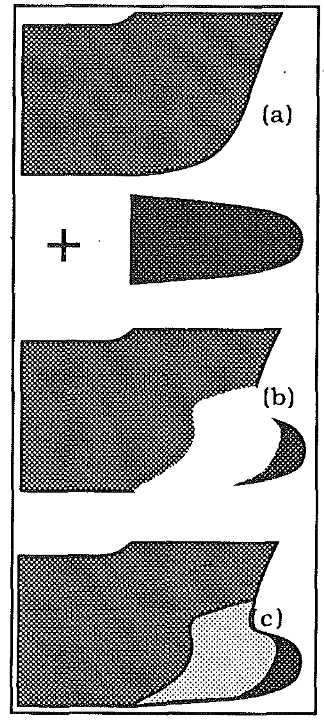


Fig. 8 An example of second order blending problem in a ship hull and bow design.

this requirement is redundant. It is indeed sufficient that only one asymptotic direction agrees, provided this direction is not tangent to the linkage curve.

- **The second fundamental tensors of the two patches are identical at all points of the linkage curve.** By this we mean that the tensors agree as basis independent multilinear maps. Their matrix representations agrees only when both tensors are expressed in the same basis [5].
- **The principal directions of curvature and the principal radii of curvature agree at all point of the linkage curve [6].**

In this paper we shall contribute two other equivalent formulations that are very valuable for implementation proposes.

- *Two surfaces tangent at a point* $\mathbf{p}_0$ *have the same normal curvatures if and only if their normal curvatures agree in three tangent directions, of which any pair is linearly independent* (*Three Tangents theorem*).
- *Two surfaces tangent along a* C^1*-smooth linkage curve are curvature continuous if and only if, at every point of the linkage curve, their normal curvature agrees for a direction other than the tangent to the linkage curve* (*Linkage Curve theorem*). *This result is very valuable for practical purposes for two reasons. Firstly it reduces the dimensionality of the problem to that of curve fairing. Secondly, the linkage curve needs only be first order continuous. This result is rather amazing for we can obtain a* C^2*-smooth blend from a* C^1*-smooth linkage curve. It is also advantageous from a practical viewpoint, for it allows the designer to use lower order piecewise parametric curves to draw the linkage curve.*

3 Related Works

The problem being considered in second order blending of functional surfaces can be formulated in the following way (Fig. 8).

Given two functional surfaces; for example a ship hull and a bow, Fig. 8(*a*), construct a boolean union of those surfaces by:

(*i*) Trimming excess material along the desired linkage curves on hull and bow, Fig. 8(*b*).

(*ii*) Joining the hull and bow by a blending surface patch such that the resulting surface is second order smooth, Fig. 8(*c*).

Without going into the details of the implementation, Section 6 will demonstrate the practicality of the Linkage Curve theorem of Section 5 in resolving the above problem.

The literature on second order continuous blending of surfaces is quite sparse. Ricci [7] proposed a method that performs both constructive solid geometry and blending of algebraic surfaces in one operation. Ricci's method is simple and efficient; however, it has a major disadvantage for it modifies globally the initial geometry, i.e., the hull and bow are no longer the surfaces initially designed.

There is a large body of work in curve fairing [4] and in adaptive first order blending of surfaces [8,9] that do not alter the global geometry. Work in curvature continuous surface blending that does not alter the global geometry is very sparse and essentially follows three trends: blending of contiguous parametric patches, blending of algebraic surfaces with algebraic surfaces, interactive trimming and blending with lofted fourth order B-splines.

As early as 1976, Veron et al. [6] formulated the curvature continuity conditions for contiguous biparametric surfaces expressed by polynomial tensor products. Using the principal directions and radii of curvature criteria, Veron et al. derived two scalar equations for curvature continuity. They showed that one of these equations derived from normal continuity, hence that only one equation needed to be satisfied. Interestingly, the first equation translates the fact that normal curvatures in the direction tangent to the linkage curve of two normal continuous patches agree. This fact was formally established by Pegna [1] and is used in the proof the linkage curve theorem in Section 5. The second equation is a direct application of the linkage curve theorem of Section 5 to the case of contiguous polynomial tensor product patches.

Kahmann [3] formulated the second order continuity problem by requiring that the surface patches be normal continuous and that the asymptotic directions of both surface patches coincide at every point of their common boundary. (The asymptotic directions are the tangent directions, real or complex, along which the normal curvature is null). In the light of the linkage curve theorem and its corollary proved in Section 5, it appears that Kahmann's formulation is redundant and that it is sufficient that one asymptotic direction coincides, as long as it is not tangent to the linkage curve. Kahmann's method may not be easily applicable for design purposes because asymptotic directions are complex when the Gaussian curvature is positive. Furthermore, Kahmann's work is specific to Bézier patches connected at their boundary.

Hopcroft and Hoffmann [10] introduced an elegant method for an adjustable first order blending surface that does not alter the original design of the surfaces. Hopcroft and Hoffmann's method applies only to polynomial surfaces in implicit form, that is, with equation $G(x,y,z)=0$ and $H(x,y,z)=0$, where G and H are polynomials. The intersection of equipotential surfaces of equations $G-t=0$ and $H-s=0$ defines a curve $S(G-t, H-s)$. When s and t are related by $f(s,t)=0$, then $f(H,G)=0$ becomes the equation of a surface which is the result of a nonrigid sweep of the intersection of equipotential surfaces, thus mapping one linkage curve into another. The boundary conditions that f must satisfy for the blending surface to be tangent to the primitive surfaces are given by theorem 2.1 in [9]. Hoffmann and Hopcroft proved that the family of intersection curves that generates a blending surface can be extended to $S(G-sW, H-tW)$ where W is any polynomial of degree less than the least degree of H or G. Relying on this argument, Hoffmann and Hopcroft argue that higher order blending could be attained. However, the degree of the blending surface goes up very rapidly to become difficult for most technical purposes. Furthermore, the method requires that the original surfaces are given by implicit algebraic equations, which does not appear often in engineering design.

The first real breakthrough in interactive second order blending is due to Hansmann [2], who introduced the definition of linkage curve in blending that we use in this paper. An accurate description of this curve is required for interactive design of the blend. Hansmann designs interactively a linkage curve γ as fourth-order B-splines parameterized in t in the parametric domain of a patch Σ. Note that the linkage curves are fourth order B-splines in the parametric domain, but not necessarily in the geometric domain. For example, if the patch Σ is a 4×4 B-spline surface, the resulting geometric curve Γ lies on the surface patch Σ and is a 19th order B-spline curve. If an initial surface is not representable as a tensor product polynomial (e.g., a cylinder) then the linkage curve is not necessarily a tensor product. Blending of the two surfaces uses a map from one linkage curve to the other. The map between the linkage curves is produced by relating isoparametric values on the two linkage curves. The blending patch is then lofted with a fourth order B-spline, orthogonal to the linkage curves at both ends and such that the first and second derivatives at the endpoints are equal respectively to the first and second parametric derivatives on the surface in the direction orthogonal to the linkage curve's tangent. In the course of our investigation, it turned out that Hansmann's claim of achieving second order continuity is only substantiated by the linkage curve theorem established here in Section 5. Hansmann's work resulted in a robust method for C^2 continuous blending. However the definition of the map between independently designed linkage curves by isoparametric values may introduce a twist in the blending patch. Also, the definition of the linkage curve as a parametric polynomial in the parametric domain may lead to high order blending surface patches.

4 Three Tangents Theorem

In Section 2 it was stressed that the following is possible: two surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$ may be tangent at a point $\mathbf{p}_0$. They may have common normal curvatures along two linearly independent tangent directions $\mathbf{t}_1$ and $\mathbf{t}_2$, and yet they may not have the same normal curvatures at the point $\mathbf{p}_0$ for all tangent directions. We shall prove in this section that the situation is different if the two surfaces have common normal curvature along three tangent directions, of which any sub-couple of directions are linearly independent.

Three Tangents Theorem: *Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be two* C^2*-smooth surfaces that are tangent at a point* $\mathbf{p}_0$*. The two surfaces have the same normal curvatures along any tangent direction at that point* <u>*if and only if*</u> *they have the same normal curvatures along three tangent directions* $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3$ *of which any two are linearly independent.*

<u>Proof:</u> We shall prove only the "if" part of the theorem, for its converse is trivial. Without loss of generality one can assume that the common tangent plane to $\mathbf{S}_1$ and $\mathbf{S}_2$ at point $\mathbf{p}_0$ is the (x,y) plane. (One can always make this claim true by an appropriate change of basis). Let $(x_i, y_i)^T$ be the coordinates of the tangent vector $\mathbf{t}_i$ $(i=1,2,3)$.

Let

$$g_\alpha = \begin{bmatrix} E_\alpha & F_\alpha \\ F_\alpha & G_\alpha \end{bmatrix} \text{ and } b_\alpha = \begin{bmatrix} L_\alpha & M_\alpha \\ M_\alpha & N_\alpha \end{bmatrix}$$

be respectively the first and second fundamental tensors of $\mathbf{S}_\alpha$ at point $\mathbf{p}_0 (\alpha=1, 2)$.

The normal curvatures along the tangent directions $\mathbf{t}_i$ are given by equation (6). Hence we obtain a system of three equations $(i=1,2,3)$.

$$\frac{L_1 x_i^2+2M_1 x_i y_i+N_1 y_i^2}{E_1 x_i^2+2F_1 x_i y_i+G_1 y_i^2} = \frac{L_2 x_i^2+2M_2 x_i y_i+N_2 y_i^2}{E_2 x_i^2+2F_2 x_i y_i+G_2 y_i^2} \quad (10)$$

Because $\mathbf{S}_1$ and $\mathbf{S}_2$ are tangent at $\mathbf{p}_0$, the first fundamental tensors $\mathbf{g}_1$ and $\mathbf{g}_2$ are equal (see Assertion 1, Section 5). Equations (10) then simplify into a system of linear equations

$$L_1 x_i^2+2M_1 x_i y_i+N_1 y_i^2 = L_2 x_i^2+2M_2 x_i y_i+N_2 y_i^2, \quad (11)$$

which can be rewritten in matrix form:

$$\mathcal{L} \begin{bmatrix} L_1 \\ M_1 \\ N_1 \end{bmatrix} = \mathcal{L} \begin{bmatrix} L_2 \\ M_2 \\ N_2 \end{bmatrix} \quad (12)$$

with $$\mathcal{L} = \begin{bmatrix} x_1^2 & 2\,x_1\,y_1 & y_1^2 \\ x_2^2 & 2\,x_2\,y_2 & y_2^2 \\ x_3^2 & 2\,x_3\,y_3 & y_3^2 \end{bmatrix}.$$

Equation (12) implies that the second fundamental tensors of the two surfaces are equal unless the matrix $\mathcal{L}$ is singular. But we shall now prove that $\mathcal{L}$ cannot be singular under the assumptions of the three tangents theorem. Without loss of generality, one can assume $(x_1, y_1)^T = (1, 0)^T$. (One can always make this claim true by an appropriate change of basis). Then the determinant of the matrix $\mathcal{L}$ is:

$$|\mathcal{L}| = 2\,y_2\,y_3(x_2\,y_3 - y_2\,x_3). \qquad (13)$$

Neither y_2, y_3 nor $(x_2\,y_3 - y_2\,x_3)$ can vanish for it would violate our assumption that $(\mathbf{t}_1, \mathbf{t}_2)$, $(\mathbf{t}_1, \mathbf{t}_3)$ and $(\mathbf{t}_2, \mathbf{t}_3)$ are linearly independent. Hence the determinant cannot be zero and the matrix $\mathcal{L}$ is nonsingular, which proves the theorem. ■

5 Linkage Curve Theorem

In Section 4, we showed that the problem of proving that two tangent surfaces have the same curvature along all tangent directions (infinite set) at a single point reduces to showing that the normal curvatures agree along three independent directions at that point. This is very advantageous for shape interrogation in computer aided design. However, the three tangents theorem is not adapted to the design of curvature continuous blends, for one would have to check that the conditions on three normal curvatures are met at every points of the linkage curve.

We shall prove below that the situation becomes different if we require that the surfaces $\mathbf{S}_1$, $\mathbf{S}_2$ be tangent along a C^1-smooth arc instead of being tangent only at a single point. In order to formulate the Linkage Curve theorem we introduce first some notations and assumptions. We shall use these notations and assumptions later on several times.

Let $\mathbf{F}$ and $\mathbf{G}$ be two regular C^2-smooth surface pieces which are represented by parameterizations $f(s,t)$, $g(u, v)$ respectively. This means f: $\mathbf{O} \to \mathcal{R}^3$ and $\mathbf{g}$: $\mathbf{A} \to \mathcal{R}^3$ are C^2-smooth maps (functions) defined on open subsets $\mathbf{O}$ and $\mathbf{A}$ of $\mathcal{R}^2$, with $f(\mathbf{O}) = \mathbf{F}$ and $g(\mathbf{A}) = \mathbf{G}$. Let $\mathfrak{I} \subset \mathcal{R}$ be an open real interval and let $\mathcal{C}$: $\mathfrak{I} \to \mathbf{F} \cap \mathbf{G}$ be a C^1-smooth path with no stationary points, i.e.

$$\mathcal{C}'(t) \neq 0 \text{ for all } t \in \mathfrak{I},\ \mathfrak{I} \neq \emptyset,\ \mathcal{C}'(t) = \frac{d}{dt}\,\mathcal{C}(t) \qquad (14).$$

Linkage Curve Theorem: Assume that both surfaces $\mathbf{F}$ *and* $\mathbf{G}$ *are tangent along* $\mathcal{C}(\mathfrak{I})$. *Then the following holds:*

(*a*) *If at some given point* $\mathcal{C}(t_0) \in \mathcal{C}(\mathfrak{I})$ *for some tangent direction* X *linearly independent from* $\mathcal{C}'(t_0)$ *the normal curvatures of* $\mathbf{F}$ *and* $\mathbf{G}$ *agree, then the normal curvatures of* $\mathbf{F}$ *and* $\mathbf{G}$ *agree for all tangent directions at the point* $\mathcal{C}(t_0)$.

Further we have:

(*b*) *We can represent both surfaces locally in a neighborhood of* $\mathcal{C}(t_0)$ *as graphs over an x,y-plane where the z-axis is chosen parallel to the common normal of the surfaces* $\mathbf{F}$ *and* $\mathbf{G}$ *at* $\mathcal{C}(t_0)$. *We have* C^2*-smooth height functions* $\tilde{f}(x,y)$, $\tilde{g}(x,y)$ *and there exist an open neighborhood* $\mathcal{V} \subset \mathcal{R}^3$ *of* $\mathcal{C}(t_0) = (x_0,y_0,z_0)$ *and an open neighborhood* $\mathcal{U} \subset \mathcal{R}^2$ *of* (x_0,y_0) *such that the points* $(x,y,\tilde{f}(x,y))$ *and* $(x,y,\tilde{g}(x,y))$ *belong to* $\mathcal{V}$ *for all* (x,y) *in* $\mathcal{U}$. *The first derivatives of* $\tilde{f}$ *and* $\tilde{g}$ *agree at the projection* $(x(t),y(t))$ *of the point* $\mathcal{C}(t) = (x(t),y(t),z(t))^T$.

The preceding statements can be formally rephrased as follows:

$$\left.\begin{array}{c} \mathbf{F} \cap \mathcal{V} = \{(x,y,\tilde{f}(x,y))\,|\,(x,y) \in \mathcal{U}\} \\ \text{and} \\ \mathbf{G} \cap \mathcal{V} = \{(x,y,\tilde{g}(x,y))\,|\,(x,y) \in \mathcal{U}\} \\ \text{and} \\ (\partial_y \tilde{f})(x(t),y(t)) = (\partial_x \tilde{g})(x(t),y(t)) \\ \text{and} \\ (\partial_y \tilde{f})(x(t),y(t)) = (\partial_y \tilde{g})(x(t),y(t)) \end{array}\right\} \qquad (15)$$

Now our main claim is: at the point $\mathcal{C}(t_0) = (x(t_0),y(t_0), \tilde{f}(x(t_0,y(to)))^T$ *we have:*

$$\begin{pmatrix} \tilde{f}_{,xx} & \tilde{f}_{,xy} \\ \tilde{f}_{,yx} & \tilde{f}_{,yy} \end{pmatrix} = \begin{pmatrix} \tilde{g}_{,xx} & \tilde{g}_{,xy} \\ \tilde{g}_{,yx} & \tilde{g}_{,yy} \end{pmatrix}. \qquad (16)$$

(*c*) *If* (*a*) *holds for any other point* $\mathcal{C}(\tilde{t}) = (x(\tilde{t}),\ y(\tilde{t}),\ z(\tilde{t}))$ *with* $(x(\tilde{t}),y(\tilde{t})) \in \mathcal{U}$ *then* (*16*) *is also valid there.*

This last claim (c) was introduced to show that there is no need to reparametrize at all points of $\mathcal{U}$ for the theorem to hold true at any point $\mathcal{C}(t)$ in a neighborhood of $\mathcal{C}(t_0)$ if one has already performed the reparametrization in the neighborhood of $\mathcal{C}(t_0)$. Now, for the proof of the theorem we shall make use of several assertions.

Assertion 1: Let $\mathbf{F}$ and $\mathbf{G}$ be as above and assume that $\mathbf{F}$ and $\mathbf{G}$ are tangent at a point $\mathbf{p}_0$. That is, $\mathbf{p}_0 = f(s_0,t_0) = g(u_0,v_0)$ and

$$\text{span}(f_{,s}, f_{,t}) = \text{span}(g_{,u}, g_{,v}) \text{ at } \mathbf{p}_0; \qquad (17)$$

where $f_{,s} = \partial f/\partial s$ etc. and span $(f_{,s}, f_{,t})$ is the set of linear combination of the vectors $f_{,s}$ and $f_{,t}$.

We can rephrase (17) by saying the images of the two maps $Df = (f_{,s}, f_{,t})$, and $Dg = (g_{,u}, g_{,v})$: $\mathcal{R}^2 \to \mathcal{R}^3$ agree. This holds still if we make reparameterizations of $\mathbf{F}$ and $\mathbf{G}$, say using diffeomorphisms (parameter transformations) ϕ, ψ

$$(s,t) = \phi(x,y),\ (u,v) = \psi(\tilde{x},\tilde{y}).$$

In this case the images of $(Df \circ D\phi)$ and $(Dg \circ D\psi)$: $\mathcal{R}^2 \to \mathcal{R}^3$ agree if we evaluate the derivatives $D\phi = (\phi_{,x}, \phi_{,y})$ and $D\psi = (\psi_{,\tilde{x}}, \psi_{,\tilde{y}})$ at $\phi^{-1}(s_0,t_0)$ and $\psi^{-1}(u_0, v_0)$ respectively.

This holds because $D\phi(\mathcal{R}^2) = D\psi(\mathcal{R}^2) = \mathcal{R}^2$ and because of (17).

Note:

$$\mathcal{R}^2 \xrightarrow[D\psi]{D\phi} \mathcal{R}^2 \xrightarrow[Dg]{Df} \mathcal{R}^3 \qquad (18)$$

Assertion 2: Let $f(s_0,t_0)$ be any point with $(s_0,t_0) \in \mathbf{O}$. Then there exists a neighborhood $\mathcal{V}$ of $f(s_0,t_0)$ and we can represent $\mathbf{F} \cap \mathcal{V}$ locally as a graph over an x,y-plane with a C^2-smooth height function $z = \tilde{f}(x,y)$, the z axis being parallel to the surface normal $\mathbf{n}$ of $\mathbf{F}$ at $f(s_0,t_0) = (x_0,y_0,z_0)$, i.e.

$$\vec{\mathbf{n}}(s_0,t_0) = \frac{(\partial_s f)(s_0,t_0) \wedge (\partial_t f)(s_0,t_0)}{\|(\partial_s f)(s_0,t_0) \wedge (\partial_t f)(s_0,t_0)\|}. \qquad (19)$$

Further we have a local reparametrization, i.e., a C^2-smooth diffeomorphism $\phi(x,y) = (s,t)$ of a neighborhood $\mathcal{U}$ of (x_0,y_0) to a neighborhood of (s_0,t_0) such that $f \circ \phi(x,y) = (x,y,\tilde{f}(x,y))^T$ for all $x,y \in \mathcal{U}$.

Proof of Assertion 2: Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthonormal vectors with $\mathbf{e}_3 = \mathbf{n}\ (s_0,t_0)$. Choosing $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ as directions for the x,y,z-axes respectively. We get for the coordinates of the surface $\mathbf{F}$

$$\begin{aligned} x(s,t) &= \langle f(s,t),\mathbf{e}_1 \rangle \\ y(s,t) &= \langle f(s,t),\mathbf{e}_2 \rangle \\ z(s,t) &= \langle f(s,t),\mathbf{e}_3 \rangle = \langle f(s,t),\mathbf{n}(s_0,t_0) \rangle. \end{aligned} \qquad (20)$$

Thus,

$$\begin{aligned}(\partial_s z)(s_0,t_0) &= \langle (\partial_s f)(s_0,t_0), \mathbf{n}(s_0,t_0) \rangle \\ &= \left\langle (\partial_s f(s_0,t_0), \frac{(\partial_s f)(s_0,t_0) \wedge (\partial_t f)(s_0,t_0)}{\|(\partial_s f)(s_0,t_0) \wedge (\partial_t f)(s_0,t_0)\|} \right\rangle \\ &= 0 \end{aligned} \tag{21}$$

and

$$(\partial_t z)(s_0,t_0) = 0 \quad \text{as well.}$$

Hence,

$$(\partial_s f)(s_0,t_0) = \begin{pmatrix} x_{,s}(s_0,t_0) \\ y_{,s}(s_0,t_0) \\ z_{,s}(s_0,t_0) \end{pmatrix} = \begin{pmatrix} x_{,s}(s_0,t_0) \\ y_{,s}(s_0,t_0) \\ 0 \end{pmatrix}.$$

$$(\partial_t f)(s_0,t_0) = \begin{pmatrix} x_{,t}(s_0,t_0) \\ y_{,t}(s_0,t_0) \\ z_{,t}(s_0,t_0) \end{pmatrix} = \begin{pmatrix} x_{,t}(s_0,t_0) \\ y_{,t}(s_0,t_0) \\ \emptyset \end{pmatrix} \tag{22}$$

Now, since f is regular, $(\partial_s f)(s_0,t_0)$ and $(\partial_t f)(s_0,t_0)$ must be linearly independent.

Therefore $\begin{pmatrix} x_{,s}(s_0,t_0) \\ y_{,s}(s_0,t_0) \end{pmatrix}$, $\begin{pmatrix} x_{,t}(s_0,t_0) \\ y_{,t}(s_0,t_0) \end{pmatrix}$ must be linearly independent.

Hence Det $\begin{pmatrix} x_{,s}(s_0,t_0) & x_{,t}(s_0,t_0) \\ y_{,s}(s_0,t_0) & y_{,t}(s_0,t_0) \end{pmatrix} \neq 0$, that is the Jacobian determinant Det $(D\binom{x}{y})$ does not vanish.

$$\mathrm{Det}(D\tbinom{x}{y}) \neq 0 \tag{23}$$

Recall now the inverse function theorem [11], which states that, if the derivative of a C^k-smooth function α: $\mathbb{R}^n \to \mathbb{R}^n$ is nonsingular at a point $\alpha(u)$, then α has a C^k-smooth inverse in a neighborhood of $\alpha(u)$. Using (23) in combination with the inverse function theorem, we find that there exists a C^2 smooth map (function) Φ defined in a neighborhood $\mathcal{U}$ of $(x_0,y_0) = (x(s_0,t_0), y(s_0,t_0))$ and this map (function) Φ is there the inverse of the map $(x(s,t), y(s,t))$, i.e., there

$$\Phi(x(s,t), y(s,t)) = (s,t).$$

If we define now $\tilde{f}(x,y) = z(f(\Phi(x,y)))$ then the claims of assertion 2 are obvious.

Assertion 3: Combining Assertion 1 and Assertion 2, we can find an open neighborhood $\mathcal{V}$ of $\mathcal{C}(t_0) = (x_0, y_0, z_0)$ and an open neighborhood $\mathcal{U}$ of (x_0, y_0) and two C^2 smooth height functions $\tilde{f}$ and $\tilde{g}$ such that

$$\begin{aligned} &\mathbf{F} \cap \mathcal{V} = \{(x,y,\tilde{f}(x,y)) \mid (x,y) \in \mathcal{U}\} \\ \text{and} \quad &\mathbf{G} \cap \mathcal{V} = \{(x,y,\tilde{g}(x,y)) \mid (x,y) \in \mathcal{U}\} \end{aligned}, \tag{24}$$

and for every point $(x,y,z) \in \mathbf{F} \cap \mathbf{G} \cap \mathcal{V}$ where $\mathbf{F}$ and $\mathbf{G}$ are tangent we have:

$$\mathrm{span}\left(\partial_y \begin{pmatrix} x \\ y \\ \tilde{f} \end{pmatrix}, \partial_y \begin{pmatrix} x \\ y \\ \tilde{f} \end{pmatrix}\right) = \mathrm{span}\left(\partial_x \begin{pmatrix} x \\ y \\ \tilde{g} \end{pmatrix}, \partial_y \begin{pmatrix} x \\ y \\ \tilde{g} \end{pmatrix}\right). \tag{25}$$

Now (25) yields

$$\mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \\ \partial_y \tilde{f} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \partial_y \tilde{f} \end{pmatrix}\right) = \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \\ \partial_x \tilde{g} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \partial_y \tilde{g} \end{pmatrix}\right). \tag{26}$$

Equation (26) implies that there exist scalars $\alpha, \beta \in \mathbb{R}$ such that:

$$\alpha \begin{pmatrix} 1 \\ 0 \\ \tilde{f}_{,x} \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ \tilde{f}_{,y} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \tilde{g}_{,x} \end{pmatrix}. \tag{27}$$

Clearly (27) yields that $\beta = 0$. The latter and (27) imply that $\alpha = 1$. Hence,

$$\tilde{f}_{,x} = \tilde{g}_{,x}. \tag{28}$$

In the same way it can be shown that

$$\tilde{f}_{,y} = \tilde{g}_{,y}. \tag{29}$$

Proof of the theorem: We prove part (b) of the theorem. Part (a) is an immediate consequence of Part (b) because normal curvatures at a point can be expressed by the first and second derivatives for a surface piece at that point. The first part of (b) is now an obvious consequence of Assertion 3. To prove (b) completely, it remains to show (16). For this consider a subpath $\mathcal{C}(\tilde{\mathcal{J}})$ of $\mathcal{C}(\mathcal{J})$ in a neighborhood of $\mathcal{C}(t_0)$ and let

$$\mathcal{C}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Since $\mathcal{C}(t)$ is C^1-smooth all components are C^1-smooth, as projections are C^∞-smooth. Hence,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \text{ is a } C^1\text{-smooth curve contained in } \mathcal{U} \text{ in the x,y plane.} \tag{30}$$

Now $\tilde{f}$, $\tilde{g}$: $\mathcal{U} \to \mathbb{R}$ are C^2-smooth functions. Hence the partial derivatives $\tilde{f}_{,x}$, $\tilde{g}_{,x}$, $\tilde{f}_{,y}$ and $\tilde{g}_{,y}$ are C^1-smooth functions. Therefore, we get with (30) that

$$\tilde{f}_{,x}(x(t),y(t)),\ \tilde{g}_{,x}(x(t),y(t)),\ \tilde{f}_{,y}(x(t),y(t)),\ \tilde{g}_{,y}(x(t),y(t))$$

are C^1-smooth functions defined on $\tilde{\mathcal{J}}$. (31)

We use (31) and exploit from the first part of (b) that

$$\tilde{f}_{,x}(x(t),y(t)) = \tilde{g}_{,x}(x(t),y(t)) \tag{32}$$

$$\tilde{f}_{,y}(x(t),y(t)) = \tilde{g}_{,y}(x(t),y(t)) \tag{33}$$

for $t \in \tilde{\mathcal{J}}$. Differentiating (32) and (33), we get:

$$\frac{d}{dt}\tilde{f}_{,x}(x(t),y(t)) = (\tilde{f}_{,xx}\ \tilde{f}_{,xy})\tbinom{x'}{y'} = (\tilde{g}_{,xx}\ \tilde{g}_{,xy})\tbinom{x'}{y'} \tag{34}$$

$$\frac{d}{dt}\tilde{f}_{,y}(x(t),y(t)) = (\tilde{f}_{,yx}\ \tilde{f}_{,yy})\tbinom{x'}{y'} = (\tilde{g}_{,yx}\ \tilde{g}_{,yy})\tbinom{x'}{y'} \tag{35}$$

$$= (\tilde{f}_{,xy}\ \tilde{f}_{,yy})\tbinom{x'}{y'} = (\tilde{g}_{,xy}\ \tilde{g}_{,yy})\tbinom{x'}{y'}, \tag{36}$$

where $x' = d(x_{(t)})/dt$ and $y' = d(y_{(t)})/dt$, and the second derivatives are evaluated at the point $(x(t),y(t))$.

Now (34) and (36) yield the following equations:

$$x'\tilde{f}_{,xx} + y'\tilde{f}_{,xy} + 0 = x'\tilde{g}_{,xx} + y'\tilde{g}_{,xy} + 0 \tag{37}$$

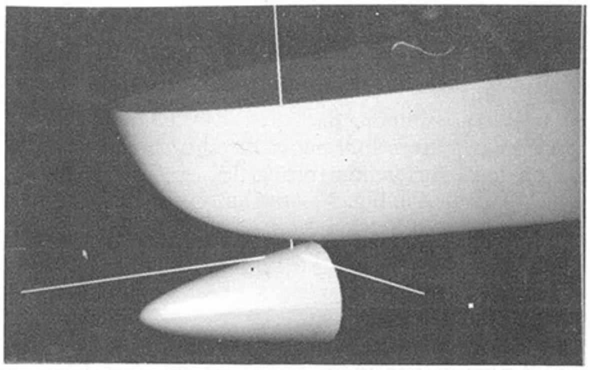
$$0 + x'\tilde{f}_{,xy} + y'\tilde{f}_{,yy} = 0 + x'\tilde{g}_{,xy} + y'\tilde{g}_{,yy}. \tag{38}$$

We shall exploit now that the normal curvatures on both surfaces $\mathbf{F}$ and $\mathbf{G}$ agree at $\mathbf{p}_0 = \mathcal{C}(t_0) = (x_0,y_0,z_0)^T$ for direction $(\tilde{x},\tilde{y})^T$ linearly independent of $(x'(t_0),y'(t_0))^T$. For this let

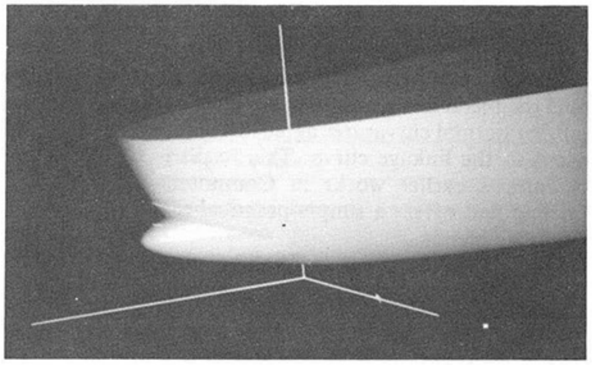
$$\tilde{\tilde{f}} = \begin{pmatrix} x \\ y \\ \tilde{f}_{(x,y)} \end{pmatrix} \text{ and } \tilde{\tilde{g}} = \begin{pmatrix} x \\ y \\ \tilde{g}_{(x,y)} \end{pmatrix} \tag{39}$$

and let $D\tilde{\tilde{f}}$, $D\tilde{\tilde{g}}$ be the differentials of $\tilde{\tilde{f}}$, $\tilde{\tilde{g}}$ respectively.

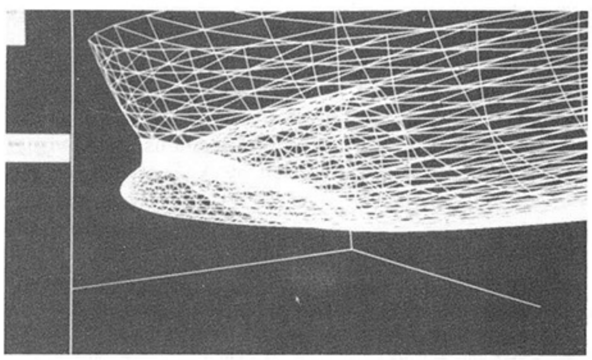
Then the normal curvature $\kappa_{\tilde{f}(x_0,y_0)}(\tilde{x},\tilde{y})$ for the surface described by $\tilde{f}$ at $\mathbf{p}_0$ for the direction $(\tilde{x},\tilde{y})^T$ is



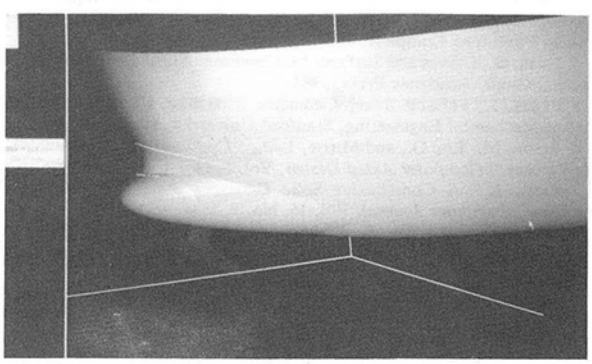
(a) The ship hull is a bicubic Bezier patch and the bow is an elliptic paraboloid given by its implicit equation.



(b) The bow is in position and trimming lines are designed interactively.



(c) A close up view of the blend in triangulated wireframe.



(d) The resulting blend with untrimmed original surfaces.

Fig. 9 The ship hull and bow blending problem.

$$\frac{(\tilde{x}^2\tilde{f}_{,xx}(x_0,y_0)+2\,\tilde{x}\tilde{y}\tilde{f}_{,xy}(x_0,y_0)+\tilde{y}^2\tilde{f}_{,yy}(x_0,y_0))}{\sqrt{1+(\tilde{f}_{,x}(x_0,y_0)^2+(\tilde{f}_{,y}(x_0,y_0)^2}\;\|(D\tilde{\tilde{f}}(x_0,y_0))(\begin{smallmatrix}\tilde{x}\\ \tilde{y}\end{smallmatrix})\|^2} \tag{40}$$

The normal curvature $\kappa_{\tilde{\tilde{g}}(x_0,y_0)}(\tilde{x},\tilde{y})$ for the surface described by $\tilde{\tilde{g}}$ at $\mathbf{p}_0$ for the direction $(\tilde{x},\tilde{y})^T$ is

$$\frac{(\tilde{x}^2\tilde{g}_{,xx}(x_0,y_0)+2\,\tilde{x}\tilde{y}\tilde{g}_{,xy}(x_0,y_0)+\tilde{y}^2\tilde{g}_{,yy}(x_0,y_0))}{\sqrt{1+(\tilde{g}_{,x}(x_0,y_0)^2+(\tilde{g}_{,y}(x_0,y_0)^2}\;\|(D\tilde{\tilde{g}}(x_0,y_0)(\begin{smallmatrix}\tilde{x}\\ \tilde{y}\end{smallmatrix})\|^2} \tag{41}$$

By assumption, the normal curvatures on **F** and **G** agree at $\mathbf{p}_0$ for a direction $(\tilde{x},\tilde{y})^T$, that is:

$$\kappa\tilde{\tilde{f}}_{(x_0,y_0)}(\tilde{x},\tilde{y})=\kappa\tilde{\tilde{g}}_{(x_0,y_0)}(\tilde{x},\tilde{y}) \tag{42}$$

The formulae (40) and (41) are straight forward consequences of the normal curvature formula (6) for surfaces given as graphs of height functions in (39). We use again that by Assertion 3

$$\tilde{f}_{,x}(x_0,y_0)=\tilde{g}_{,x}(x_0,y_0) \text{ and } \tilde{f}_{,y}(x_0,y_0)=\tilde{g}_{,y}(x_0,y_0). \tag{43}$$

Noting also that

$$D\tilde{\tilde{f}}(x_0,y_0)=\begin{pmatrix}1 & 0\\ 0 & 1\\ \tilde{f}_{,x}(x_0,y_0 & \tilde{f}_{,y}(x_0,y_0)\end{pmatrix}$$

and (44)

$$D\tilde{\tilde{g}}(x_0,y_0)=\begin{pmatrix}1 & 0\\ 0 & 1\\ \tilde{g}_{,x}(x_0,y_0) & \tilde{g}_{,y}(x_0,y_0)\end{pmatrix}$$

we find

$$D\tilde{f}(x_0,y_0)=D\tilde{\tilde{g}}(x_0,y_0) \tag{45}$$

and hence

$$\|(D\tilde{\tilde{f}}(x_0,y_0))(\begin{smallmatrix}\tilde{x}\\ \tilde{y}\end{smallmatrix})\|=\|(D\tilde{\tilde{g}}(x_0,y_0))(\begin{smallmatrix}\tilde{x}\\ \tilde{y}\end{smallmatrix})\|. \tag{46}$$

Therefore finally employing (43), (46), (40), (41) equation (42) yields easily

$$\tilde{x}^2\tilde{f}_{,xx}+2\tilde{x}\tilde{y}\tilde{f}_{,xy}+\tilde{y}^2\tilde{f}_{,yy}=\tilde{x}^2\tilde{g}_{,xx}+2\tilde{x}\tilde{y}\tilde{g}_{,xy}+\tilde{y}^2\tilde{g}_{,yy}, \tag{47}$$

with the second derivatives in (47) being evaluated at (x_0,y_0). Combining the equations (47), (37) and (38) yields a matrix equation.

$$\mathcal{L}\begin{pmatrix}\tilde{f}_{,xx}(x_0,y_0)\\ \tilde{f}_{,xy}(x_0,y_0)\\ \tilde{f}_{,yy}(x_0,y_0)\end{pmatrix}=\mathcal{L}\begin{pmatrix}\tilde{g}_{,xx}(x_0,y_0)\\ \tilde{g}_{,xy}(x_0,y_0)\\ \tilde{g}_{,yy}(x_0,y_0)\end{pmatrix} \tag{48}$$

with

$$\mathcal{L}=\begin{bmatrix}\tilde{x}^2 & 2\tilde{x}\tilde{y} & \tilde{y}^2\\ x' & y' & 0\\ 0 & x' & y'\end{bmatrix}.$$

We shall show below that the matrix $\mathcal{L}$ is not singular. Hence the inverse matrix $\mathcal{L}^{-1}$ exists. Therefore, multiplying both sides of (48) with $\mathcal{L}^{-1}$ yields

$$\begin{pmatrix}\tilde{f}_{,xx}(x_0,y_0)\\ \tilde{f}_{,xy}(x_0,y_0)\\ \tilde{f}_{,yy}(x_0,y_0)\end{pmatrix}=\begin{pmatrix}\tilde{g}_{,xx}(x_0,y_0)\\ \tilde{g}_{,xy}(x_0,y_0)\\ \tilde{g}_{,yy}(x_0,y_0)\end{pmatrix} \tag{49}$$

which proves (16) and finishes the proof of (b).

It remains to be shown that the matrix $\mathcal{L}$ is not singular. For this, developing $\det(\mathcal{L})$ along the first row gives:

$$\begin{aligned}\det(\mathcal{L})&=\tilde{x}^2y'^2-2\tilde{x}\tilde{y}\,x'y'+\tilde{y}^2x'^2\\ &=(\tilde{x}y'-\tilde{y}x')^2\\ &=(\det|\begin{smallmatrix}\tilde{x} & x'\\ \tilde{y} & y'\end{smallmatrix}|)^2\end{aligned}$$

The determinant $\det|\begin{smallmatrix}\tilde{x} & x'\\ \tilde{y} & y'\end{smallmatrix}|$ is not null by the assumption that $(\begin{smallmatrix}\tilde{x}\\ \tilde{y}\end{smallmatrix})$ and $(\begin{smallmatrix}x'\\ y'\end{smallmatrix})$ are linearly independent. Hence $\det(\mathcal{L})\neq 0$. this shows that $\mathcal{L}$ is not singular. We have also shown (c) because (37) and (38) hold for all points $\mathcal{C}(t)=(x_{(t)},y_{(t)},z_{(t)})^T$ with $(x_{(t)},\,y_{(t)})\in\mathcal{U}$ and because the argument used to derive (47) required only that (a) holds for a point $\mathcal{C}(t_0)=(x_0,y_0,z_0)^T$ with $(x_0,y_0)\in\mathcal{U}$. This finishes the proof of the theorem. ■

Note that the preceding algebraic computations showing that $\mathcal{L}$ is not singular, hence invertible, remain the same if the coordinates are complex instead of real. Hence the linkage curve theorem covers and generalizes Kahmann's result for it requires weaker assumptions and shows that only one common asymptotic direction transversal to the linkage curve is sufficient to guarantee second order smoothness. Recall that an asymptotic direction is characterized by the vanishing of equation (47). For positive Gaussian curvature equation (47) can only vanish if the coordinates of the tangent direction $(\tilde{x},\tilde{y})$ are complex numbers. So even if the Gaussian curvature is positive, i.e., asymptotic directions are complex, only one asymptotic direction needs to agree for both surface patches to guarantee second order smoothness.

These consideration allow us to derive the conclusion that follows. Our specific proof method for the Linkage Curve theorem is general enough to prove the following corollary, which in the case of positive Gaussian curvature is not covered in the statement of the Linkage Curve theorem itself.

Corollary: *If two curvature continuous surface patches have a normal continuous contact along a C^1-smooth curve $\mathcal{C}$: $[0,1] \rightarrow \mathcal{R}^3$ and if they share a common asymptotic direction at a point $\mathcal{C}(t_0)$, which is linear independent of the tangent vector $\mathcal{C}'(t_0)$ over the field of complex numbers* **C**, *then all normal curvatures agree for all directions at the point $\mathcal{C}(t_0)$ for both surface patches.*

6 Discussion and Application

The main contribution of this paper is the Linkage Curve theorem (LC theorem). The LC theorem states, in essence, that for two connected surface patches that are already normal continuous to be also curvature continuous, it is sufficient that there exists one direction other than the tangent to the linkage curve along which normal curvatures agree at every point of the linkage curve. This direction need not be continuously depending on the curve parameter and the only requirement is that it not be tangent to the linkage curve. The condition on the linkage curve to be tangent continuous can be relaxed even further to encompass possible tangent discontinuities (corner points). This remark leads to the following results, which is an immediate consequence of our proof of the linkage curve theorem.

Corollary: *Assume that two regular C^2-smooth surface patches share common surface normals (or equivalently tangent planes) along a continuous curve $\alpha:[0,1] \rightarrow \mathcal{R}^3$ where the curve pieces $\alpha:[0,1/2] \rightarrow \mathcal{R}^3$ and $\alpha:[1/2,1] \rightarrow \mathcal{R}^3$ are C^1-smooth on their respective definition intervals. Assume further that the vectors $\lim_{t \le 1/2, t \rightarrow 1/2} \alpha'(t)$ and $\lim_{t \ge 1/2, t \rightarrow 1/2} \alpha'(t)$ are linearly independent. Then for both surface patches all normal curvatures agree for all directions at the point $\alpha(1/2)$.*

This last result is useful for blending spline patches at their boundary vertices, for blending patches at corner points, or to approximate curvature continuous blends along trimmed patch boundaries.

The conditions expressed by the LC theorem for second order smoothness are of extreme simplicity. The fact that curvature continuity is obtained just by matching normal curvatures in just one tangent direction is very important for implementation. It reduces the dimensionality of the problem to that of line fairing, which is a well-understood problem. Moreover, it makes second order blending extremely well suited for swept surfaces [5].

In Pegna [1], the LC theorem is combined with particular geometric constructions that define interactively a trimming line, a monotonic map from one trimming line to the other and directions in the tangent plane at all points of the trimming lines. The trimming lines are then used as linkage curves to generate curvature continuous blend surfaces that only depend on local differential properties of the surfaces. This result is illustrated in Fig. 9 by the hull and bow blending problem introduced in Section 3.

7 Conclusion

This paper presents mathematical results in differential geometry that pertain to interactive design of curvature continuous blend surfaces. The main contribution herein is the Linkage Curve theorem (LC theorem). Second order smoothness normally requires that normal curvatures agree along all tangent directions at all points of the common boundary of two patches. The LC theorem, however, shows that if the blend is already first order smooth, then it is sufficient that normal curvatures agree in a direction other than the tangent to the linkage curve. This result is significant for it substantiates earlier works in Computer Aided Geometric Modeling and offers a simple practical mean to generate second order blends.

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