Numerical Analysis of Relaxation Oscillators Based on a Differential Geometric Approach

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***Abstract*— The difficulties to analyse the state space of a special class of nonlinear electronic circuits are illustrated and a new method to treat these problems is presented. Theoretical aspects of circuit equations from a differential geometric point of view are considered and methods for solving circuit equations by means of algorithms from computational differential geometry are presented. In this paper differential geometric methods were applied to a relaxation oscillator and numerical results were achieved. We describe the behaviour of an emitter-coupled multivibrator with differential algebraic equations and compute its state space numerically.**

I. Introduction

From the work of Brayton and Moser [1] it is known that descriptive equations of electronic circuits belong to the class of differential equations on differentiable manifolds. They formulated the circuit equations with regard to coordinates as usual [1]. A few years later, Smale [2] reformulated these equations by means of modern differential geometry.
Further work was done by Matsomoto, Ishiraku and others to refine this approach for describing electrical networks (see, e.g. Mathis [3]). On the other hand Sandberg [4] and Gear [5] tried to solve the so-called "time-constant problem" of circuit simulation, which is one of the big obstacles in constructing an efficient and general purpose circuit simulator. It was emphasized by Gear [5] that circuit equations should be considered as algebro-differential equations (DAEs). That implies that one can treat the circuit's state space as a differentiable manifold and the dynamic defined on it as a differential equation system. Although the concepts of differential geometry have been known for a long time, applications to circuit analysis and numerical results were missing until recently. In this paper, we use differential geometric algorithms to explicitly compute state spaces of electronic circuits.

Of particular interest are those circuits which can be described in DAE notation and whose manifolds contain a "fold". These kind of electronic circuits provide examples for the so-called "time-constant problem" of circuit simulation, because such a fold describes discontinuous changes in several phenomena.
These circuits can also be described as singularly perturbed systems where the dynamic can be divided in two parts, namely the slow and the fast parts. The trajectories on the manifold describe the slow behaviour of the circuit until reaching a fold. At the fold edge, which we call "jump set", the circuit equations become singular and the transients of the system "jump" immediately from one part of the manifold to another, without remaining on the manifold. To model such behaviour one can perform a regularisation. Therefore, the analysis of such an electronic circuit with common circuit simulators like SPICE, sometimes require adding regularising capacitors or inductors. When simulating such circuits without regularisation, the simulation fails because there are two or more states of the node voltages or currents. With our new approach it will not be necessary to add a regularisation. Our aim is to detect whether a circuit exhibits the described behaviour and in which coordinate system it is expressed as a fold in the manifold. In this paper we use geometric algorithms to explicitly compute the state space of a relaxation oscillator so we are able to perform these jumps directly.

II. motivation

Although jumps from the stable manifold were introduced in a heuristic manner by electrical engineers many years ago, a theoretical concept of jumps in singularly perturbed systems were developed by mathematicians. These solutions are called canards; see e. g. Guckenheimer [6]. Based on this concept numerical methods are suggested in order to calculate canard solutions [7]. Our aim is not to calculate canard solutions but the jumps with a differential geometric approach. More precisely, in our examples canard solutions do not arise, because there are no saddle points on the manifold.
The focus of our work is on electronic circuits based on the positive feedback principle, i.e. whose voltage/current characteristic contains a region of negative slope or includes negative differential resistance. It is noteworthy that many digital circuits belong to these numerically problematic circuits, because they are in fact analog circuits that retain information by assuming a certain state. When the information changes very fast, transitions may occur.
In this paper our main focus lies on relaxation oscillators which form another class of these electronic circuits. One model of a relaxation oscillator is the well known *van der Pol* oscillator, that we already described in previous papers [8], [9]. In the field of electrical engineering, relaxation oscillators provide a model for multivibrators which were investigated in detail, e.g. by Ponzo and Wax [10]. However we concentrate ourselves on the so-called emitter-coupled multivibrator [11] because of its ability to produce high frequencies.

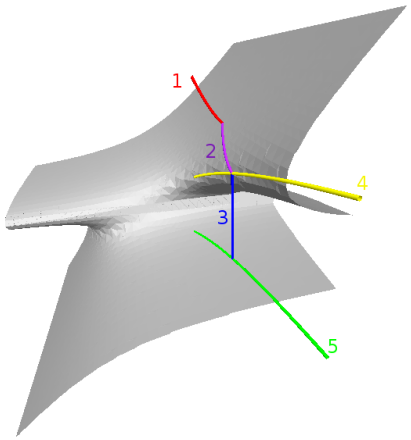

Fig. 1. The manifold defined by $z^3 - yz^2 + x = 0$. Calculated in axis-range from -5 to 5.

In our work, we use the following model for an electronic circuit:

$$\mathbf{B}(\mathbf{x})\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{y}, t), \quad (1)$$

$$0 = \mathbf{f}(\mathbf{x}, \mathbf{y}) \quad (2)$$

where $\mathbf{x}$ and $\mathbf{y}$ are the vectors corresponding to the voltages and currents, t is a time, $\mathbf{g}$ is a nonlinear vector field and $\mathbf{B}$ is a matrix related to the dynamical elements. Most circuit simulators are based on a modified nodal analysis (MNA) description which, in most cases, implies a singular matrix $\mathbf{B}$ and hence complicates the system analysis. A solution of this problem is to choose a suitable projection of the system equations to a subspace.
The state space of the electronic circuit which form the manifold is described by equation (2).

III. Computational geometric methods

Our new approach uses concepts from differential geometry to find the state space of an electronic circuit. The set of algebraic equations (2) presented above are interpreted as an implicit manifold M and the differential equations (1) describe the dynamical behaviour D of the system, which is a vector field defined on the embedding space $\mathcal{S}$.

We want to find paths on the manifold, which follow the dynamic D. To this end we use a tracing scheme: starting at a given point on the manifold we integrate the dynamic numerically and trace a path on the manifold. We make sure to stay on the manifold by projecting the dynamic to the tangential space in each step.

The first question that arises from the description of the tracing scheme, is the location of a starting point on the manifold. In our setting, the dimension k of the embedding space originates from the chosen set of electrical parameters describing the circuit. The dimension n of the manifold depends on the formulation of the model and the codimension m is simply $m = k - n$. Therefore, the searched starting point

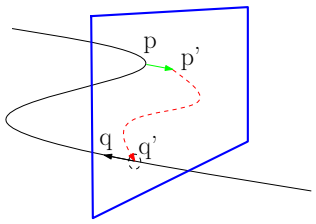

Fig. 2. Start point of the hit set.

is n-dimensional and we have a set of m implicit equations defining it.
We use a homotopy method [**?**], [12] to find a starting point on the manifold:

$$H(\mathbf{w}, \lambda) := \mathbf{f}(\mathbf{w}) + (\lambda - 1)\mathbf{f}(\mathbf{w}_0) \quad (3)$$

Here, $\mathbf{w}_0$ can be any point of the embedding space, $\mathbf{w}$ is a point on the path from $\mathbf{w}_0$ to the manifold, $\mathbf{f}(\mathbf{w})$ is an m-dimensional value stemming from the set of algebraic equations (2) and $\lambda \in [0, 1]$ is the parameter controlling the homotopy. By forcing

$$H(\mathbf{w}, \lambda) = 0 \quad (4)$$

it is obvious that for $\lambda = 0$, a solution is $\mathbf{w} = \mathbf{w}_0$ and for $\lambda = 1$, $\mathbf{f}(\mathbf{w})$ has to be zero, i.e. $\mathbf{w}$ lies on the manifold. By setting the derivative of the homotopy equation to zero, the tangential vector for the searched path is found. This makes the searched path leading to the manifold (shown in fig.1 step 1) calculable. After finding a starting point on the manifold we are interested to find the jump set respective one coordinate axis or coordinate plane, respectively.

Given a basis B_{es} of the embedding space $\mathcal{S}$, we define the *jump set* J_s of a manifold $M \subset \mathcal{S}$ respective a set $A \subset B$ of basis vectors to be the set of all points $p \in M$ whose tangent spaces $T_{\mathbf{p}}$ for which $A \cup T_{\mathbf{p}}$ is linearly dependent. The corresponding *hit set* H_s is the union of intersections of the tangent spaces of all points in J_s with M without J_s, so $H_s := \{\cup(M \cap T_{\mathbf{p}}) : \mathbf{p} \in J_s\} \backslash J_s$.

We consider a 2-dimensional manifold in R^3. Then the sign of the tangential vector is changing in the component belonging to the desired axis. We use a shooting method following geodesic curves [13] while testing the sign (see fig.1 step 2). The ending point is a start point of the jump set. From here we jump in the direction of the axis to get a start point of the hit set (see fig.1 step 3). We employ a bisection method to get any needed precision. We trace jump set and hit set together because the tangential vector needed for tracing the hit set is a projection of the tangential vector of the jump set (see fig.1 step 4 and 5). A tangential vector of a point in the jump set is calculated as cross product of the normal of the manifold and the derivation in axis direction; if we project this vector to the tangential plane of the corresponding point in the hit set, we get the corresponding tangential vector.
The principles described above have been implemented in an experimental framework in C++.

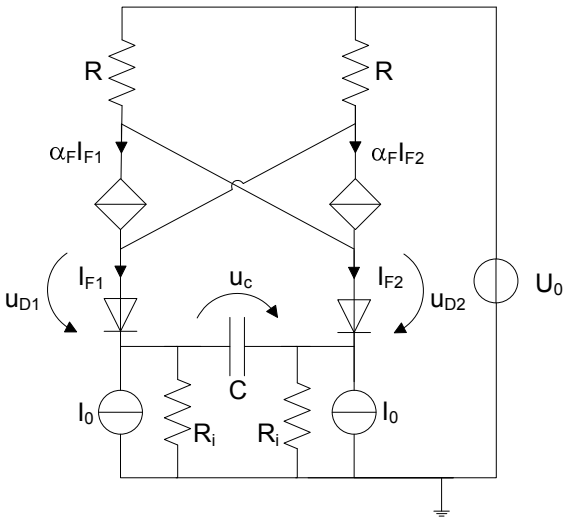

Fig. 3. Emitter-coupled multivibrator

If the codimension is higher, the situation is more difficult. Consider a one-dimensional manifold in 3-space: here the jump from $\mathbf{p} \in J_s$ to its corresponding hit point $\mathbf{q}$ is contained in a two-dimensional subspace. For this and higher-dimensional cases, we propose the following method (see fig.2): The manifold is left at $\mathbf{p} \in J_s$ in direction of the normal $\mathbf{n}_p$ for a small step to $\mathbf{p}'$. A new linear equation system is formed by intersecting the tangent space, i.e. the hyperplane spanned by the base vectors and anchored in $\mathbf{p}'$ and the manifold. Thus $\mathbf{p}$ is no solution of the new equation system and a homotopy-method starting at $\mathbf{p}'$ is forced to find another solution $\mathbf{q}'$ which lies near the hit set. We get appropriate starting-points for the homotopy-method via a bisection-method. We use this solution as initial value for a Newton bisection to compute the originally wanted hit point $\mathbf{q}$ corresponding to $\mathbf{p}$.

IV. Modelling of relaxation oscillators

To analyse the emitter-coupled multivibrator illustrated in figure 3, we use the Ebers-Moll model [11] to reasonably model the nonlinear bipolar transistors.

The dimensions are $R = 500\Omega, R_i = 100k\Omega, C = 33nF, I_S = 6.73fA, V_T = 26mV$ and $\alpha_F = 0.99$. For the simulation we use a constant bias voltage $U_0 = 5V$ and a constant bias current $I_0 = 0.26mA$.

We neglected the reverse mode, the collector-emitter and the parasitic base-emitter capacitances. Latter are usually used to enable a circuit simulation with common simulators and to analyse emitter-coupled multivibrators by the theory of singularly perturbed systems [14]. For analysing the multivibrator by the theory of singular perturbed systems one can add the parasitic capacitances $C_1 = C_2 = \epsilon << C$ parallel to the diodes D_1 and D_2 and transform the system equations to get the standard form of a three-dimensional singularly perturbed system. By setting $\epsilon = 0$, one gets the slow model which consists of one equation for the dynamic and two algebraic constraints resulting in the so-called degenerate system.

We, for our purpose, do not have to add the regularisation capacitances C_1, C_2 and by Kirchhoff's Law, we can derive the following network equations:

$$-\frac{R_i}{R}C \cdot \dot{u}_c = -\frac{u_{D1}}{R} + \left(\alpha_F - \frac{R_i}{R} - 1\right)\frac{I_s}{\alpha_F}\left(e^{\frac{u_{D1}}{26 \cdot 10^{-3}}} - 1\right) + \frac{U_0}{R} + \frac{R_i}{R}I_0 - I_s\left(e^{\frac{u_{D2}}{26 \cdot 10^{-3}}} - 1\right) \quad (5)$$

$$\frac{R_i}{R}C \cdot \dot{u}_c = -\frac{u_{D2}}{R} + \left(\alpha_F - \frac{R_i}{R} - 1\right)\frac{I_s}{\alpha_F}\left(e^{\frac{u_{D2}}{26 \cdot 10^{-3}}} - 1\right) + \frac{U_0}{R} + \frac{R_i}{R}I_0 - I_s\left(e^{\frac{u_{D1}}{26 \cdot 10^{-3}}} - 1\right) \quad (6)$$

$$2R_iC \cdot \dot{u}_c = -u_c + R_i \cdot \frac{I_s}{\alpha_F}\left(\left(e^{\frac{u_{D1}}{26 \cdot 10^{-3}}} - 1\right) - \left(e^{\frac{u_{D2}}{26 \cdot 10^{-3}}} - 1\right)\right) \quad (7)$$

In equation (5), (6) and (7), u_{D1} and u_{D2} represent the voltages at the diodes D_1 and D_2, respectively, and u_c is the voltage at the capacitor C. With appropriate algebraic manipulations (projection), we recover a semi-explicit DAE

$$\begin{aligned} 0 &= f_1(u_{D1}, u_{D2}, u_c) \\ 0 &= f_2(u_{D1}, u_{D2}, u_c) \\ \dot{u}_c &= g(u_{D1}, u_{D2}, u_c) \end{aligned} \quad (8)$$

and get a fold in the manifold shape. As we see, one of the main problems of describing electronic networks with differential geometry is to choose the correct projection. So one has to consider two main factors in projecting the system equations: a regular matrix $\mathbf{B}$ and getting a fold in the manifold. From a geometric point of view, we look for a specific coordinate system where these properties are satisfied. Nevertheless, we found a proper projection and got a DAE of the form (8), where $B(x) = 1$. Representing the state space of the relaxation oscillation, the algebraic constraints form the slow manifold given by the intersection of the solution sets of $f_1(u_{D1}, u_{D2}, u_c) = 0$ and $f_2(u_{D1}, u_{D2}, u_c) = 0$.

Figure 4 shows the two surfaces given by $f_1 = 0$ and $f_2 = 0$ and figure 5 displays the intersection of both surfaces. As one can see, the intersection is S-shaped and forms the slow manifold of the multivibrator. Figure 5 was calculated for $u_{D1} = u_{D2} = 0.2V \ldots 0.7V$ and the maximum and minimum of the curve lie between $u_c \cong -0.19V$ and $u_c \cong 0.19V$. To verify our results, we have compared the range of the state space from fig. 5 with a SPICE simulation of a regularised emitter-coupled multivibrator.

At the maximum and minimum, the slow manifold loses its stability, i.e. $det(\frac{\partial f}{\partial(u_{D1}, u_{D2})}) = 0$ with $f = [f_1, f_2]^T$. Due to the constraints, the trajectories move corresponding to the tangential component of the vector field defined in eq. (8) along the slow manifold. Just regarding the dynamic in eq. (7) without knowing that the system has an oscillation as solution, the trajectories would stop at the maximum and minimum of the slow manifold.

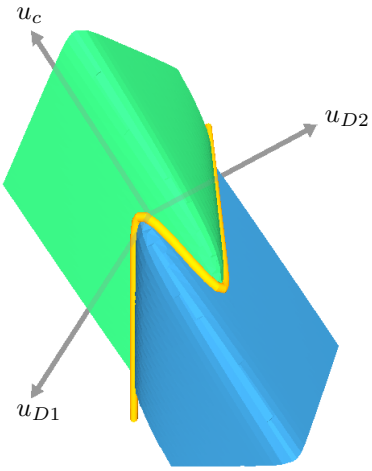

Fig. 4. Intersection of two manifolds

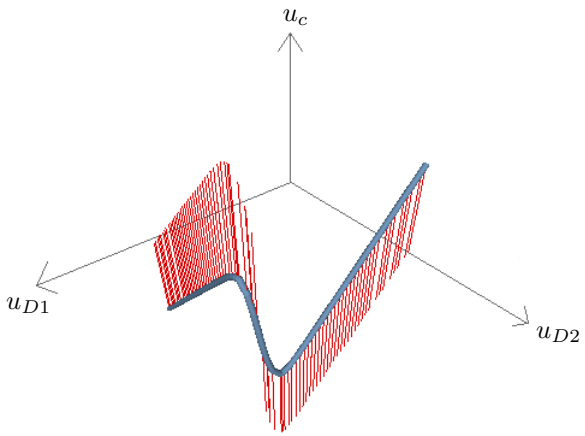

Fig. 5. Curve of the state space and its normalized dynamic

In fact, the trajectories move rapidly away and jump parallel to the plane $u_{D1} - u_{D2}$, because the voltage across the capacitance C cannot jump. With our new geometric approach we do not have to add parasitic capacitances to reproduce this behaviour. We just have to determine the jump and hit sets of the folded manifold and connect the curves.
This approach can also be used to determine the oscillation frequency of electrical multivibrator circuits. Following [15] and [14] the periodic time only depends on the slow phenomena while the duration of the jumps is neglected.

V. Conclusion

In this paper we have shown numerical results of applying differential geometry for finding the state space of a relaxation oscillator. We have verified our results with a circuit simulation of a regularised relaxation oscillator with the same design parameters. Especially we show how to explicitly calculate jump and hit sets on the state space manifold that capture the behaviour of the circuit.

A. Outlook

By now our program limits the embedding space to a dimension of 3, but a second implementation with reduced limitations is being developed. These cases are much more complicated, because the manifold can have any dimension. Also a larger codimension causes some new problems. In these cases, the jump set and hit set can have more than one dimension. Hence new techniques for describing these sets are necessary. We think about employing geodesic polar coordinates to this end. In higher dimensions, visualisations of results are not self-evident. Display facilities are needed which extract the interesting content automatically.

To successfully apply the presented methods on circuit analysis, the following steps are necessary: Based on the classical theory of nonlinear electronic circuits and differential geometry, a systematical projection for detecting the fold in the manifold has to be constructed. Additionally, the methods have to be extended to higher-dimensional spaces. With this new circuit description, unknown characteristics of the electronic circuits could be gained and used for design improvements.

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