Nonlinear Electric Circuit Analysis from a Differential Geometric Point of View

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***Abstract*—Theoretical aspects of circuit equations from a differential geometric point of view are considered and methods for solving circuit equations by means of algorithms from computational differential geometry are presented. These methods are illustrated by means of some simple circuit examples.**

I. INTRODUCTION

It is known since the early Sixties that descriptive equations of electrical circuits belong to the class of differential equations on differentiable manifolds. This result is related to the celebrated paper of Moser and Brayton [1] in 1964 where their equations for the description of reciprocal and nonlinear circuits are written in coordinates as usual. It lasted another few years until the equations of Moser and Brayton were reformulated by Smale [2] by means the framework of modern differential geometry. Further work was done by Matsomoto, Ishiraku and other to refine this approach for describing electrical networks (see e.g. Mathis [3]). On the other hand Sandberg and Gear tried to solve the so-called "time-constant problem" of circuit simulation being one of the big obstacles to construct an efficient and general purpose circuit simulator. It was emphasized by Gear that circuit equations should be considered as algebro-differential equations (DAEs) but it lasted more than another ten years until Linda Petzold - a former Ph.D. student of Gear - found out in 1982 that '"DAEs are not ODEs" (ODEs: ordinary differential equations). For references and further information see e.g. Mathis [4].

At the beginning of the Eighties – approximately twenty years after the understanding that these circuit equations are of a more general type than ordinary differential equations – it became clear that circuit equation should be considered as differential equations on differentiable manifolds or algebro-differential equations. A detailed presentation of the concept of circuit theory from the point of view of modern differential geometry is included in Robert Hermann's monographs on "Interdisciplinary Mathematics"' where the following statement is formulated: "Electrical circuits offers prototypes and examples of many sorts of abstract mathematical and physical structures; it is extremely useful and important to sort out such generalizations, since it seems that many situations - in biology, chemistry, economics and physics - can be modelled by means of these mathematical structures." (Herman [5])

II. MOTIVATION

Especially problematic for the numerical analysis of electronic circuits are non-linear electronic devices, whose functionality are based on the feedback principle or electronic devices, whose voltage/current characteristic includes a region of negative slope (negative differential resistance). Furthermore the multi-vibrators, Schmitt trigger circuits and comparator circuits should be mentioned in this context. Also many digital circuits belong to the numerically problematic circuits, because these are in fact analoge circuits that retain information by assuming a certain state. When the information changes very fast, transitions may occur. For modeling this class of electronic circuits, it will be necessary to use differential equations with singularities.

III. COMPUTATIONAL METHODS FROM DIFFERENTIAL GEOMETRY

Although concepts from differential geometry are known for a long time studying theoretical aspects of circuit analysis computational concepts based on differential geometric ideas are missing until recently. In this paper we use geometric algorithms to explicitly compute operation points for electronic circuits. As mentioned, we can treat state space as a differentiable manifold and the dynamic defined on it as a differential equation system. Opposed to conventional methods which are using homotopy methods to search operation points, here homotopy method are only used in finding specific starting points on the manifold. The principles of the homotopy methods was already used in the publications of Naß and Wolter [7] and [8]. After obtaining a starting point, we use the dynamic on the manifold to trace a solution. In this approach we will consider the derived circuit equations as a geometrical problem only. We can then trace a curve on the manifold by numerically integrating the given differential equations which describe a tangent vector field on the manifold. This should lead us to an operation point or, if an oscillating circuit is given, represents the set of states.

For many problems the manifolds are shaped as folded surfaces. These surfaces are embedding the solution curves, which may reach a maximum with degenerated dynamics.

After a "Jump" the curve may continue in a different surface area. This behaviour is described by and Shankar Sastry.[16] The above problem can be solved by a Tichonov regularisation [3], [9] which transforms the algebraic equation to a differential equation.

In the folowing chapters we will attempt to illustrate our methods with simple examples.

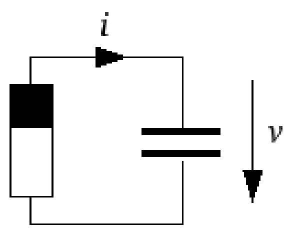

Fig. 1. The simple circuit of the Van der Pol oscillator consisting of a capacitor and a nonlinear resistor.

IV. Examples

A. Van-Der-Pol-Oscillator

The degenerated Van-der-Pol-Oscillator is a simple circuit consisting of a resistor and a capacitor that are connected in a circle, described by:

$$\frac{dv}{dt} = i \quad (1)$$

$$0 = -v - i^3 + i \quad (2)$$

where the differential equation (1) characterizes the capacity and the non-linear relation (2) defines the resistance. A diagram of this simple oscillator is shown in Figure 1. To understand this example the shape of the curve of the nonlinear resistance is important. This is shown in Figure 2.

The curve shown in Figure 2 may also be similarly considered as a manifold containing the dynamics. From this geometric point of view, the solution of (2) is a one-dimensional manifold, e. g., a curve M in the plane and the differential equation generates a dynamic which should be solved with respect to the current i. But this is not feasible globally, since in the extrema of the curve wrt. v, the dynamic degenerates to 0. Since $i \neq 0$ in these points, they can not be equilibria and therefore the model does not capture the behavior of the circuit.

The described problem can be solved by a so-called Tichonov regularisation [9], [10] which transforms the algebraic equation to a differential equation

$$\varepsilon \frac{di}{dt} = -v - i^3 + i. \quad (3)$$

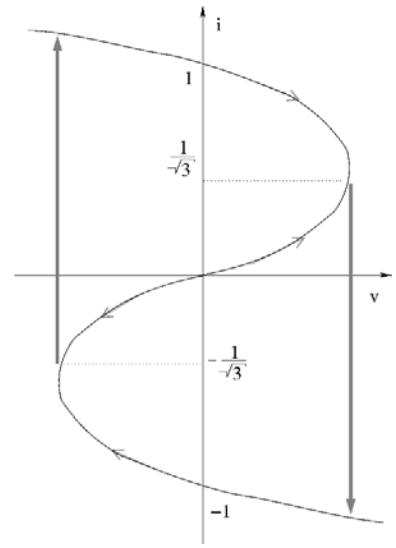

Fig. 2. The resistance curve is regarded as a manifold. Additionally the dynamic projected on the manifold has been drawn on the surface. The vertical arrows indicate the jump behaviour.

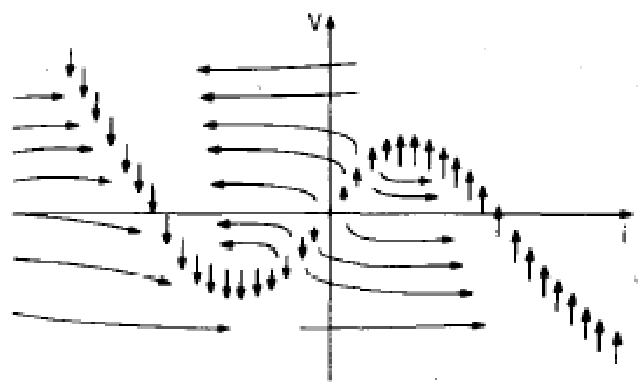

Fig. 3. The result after the Tichonov regularisation for the *Van Der Pol* example

with ε near zero. The dynamic of the system is now generically smooth and the formerly singular points exhibit a very fast dynamic, the system "jumps" from a formerly singular point tangentially to another area of the manifold. We want to capture this phenomenon with differential geometric tools and trace the curve on the manifold to an extremum where it jumps tangentially, thus following the oscillating path.

The result of the Tichonov regularization for the simple Van der Pol example is shown in Figure 3.

The jumps described above occur in a special class of folded manifolds. These are embedded in a space whose axes can be associated with unlimited voltages and currents of the circuit. Now, a geometrically interpretable mapping S assigns drop points to bounce points. the drop points are located on

Fig. 4. Possible curves jump on a folded manifold.

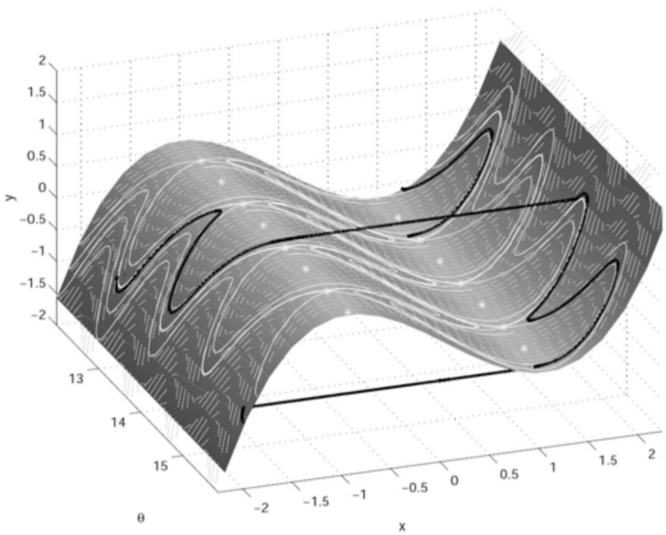

Fig. 5. Plot of the trajectory of the sinusoidally excited Van-Der-Pol-Oscillator on the critical manifold of the system. from [15].

maxima curves - the bounce points lie on a coreespondending sheet of the same manifold. Such a jump can be done in different ways.

First, the points can be mapped by tangential projection on a resulting perpendicular curve. On the other hand, the points can be mapped by extending the current tangent vector at the time of the bounce. The last option is certainly more difficult to calculate but is still possible in an acceptable time.

B. Sinusoidally Excited Van-Der-Pol-Oscillator

Another example is the sinusoidally excited Van-Der-Pol-Oscillator. (see: Guckenheimer[15]) This is a simple example where the manifold is a two-dimensional surface in three-dimensional space. The following equations are a description of the sinusoidally excited Van-Der-Pol-Oscillator:

$$\varepsilon \dot{x} = y + x - \frac{x^3}{3}, \tag{4}$$
$$\dot{y} = -x + a\sin(2\pi\Theta), \tag{5}$$
$$\dot{\Theta} = \omega, \tag{6}$$

with the corresponding vector field on the cylinder:

$$\mathbb{R}^2 \times M^1$$

is defined. The slow dynamic of the system is obtained for $\varepsilon = 0$ where derivate the side of (4) and scaling time with factor $(x^2 - 1)$ with $(d\tau := dt/(x^2 - 1))$. Resulting in the following equations:

$$\Theta' = \omega(x^2 - 1), \tag{7}$$
$$x' = -x + a\sin(2\pi\Theta) \tag{8}$$

with $(\cdot)' \equiv d/d\tau$.

In addition to the (possibly) slow dynamics figure 5 shows that the trajectories at certain points leave the manifold with a jump and reach another part of the manifold. As with the autonomous van der Pol equation is a solution to this behavior explained by using a regularization. The relationships are geometrically interpretable.

The mathematical and analytical part of the project essentially is differential-geometric studies of any state manifolds which appear by electronic circuits.

V. Conclusion

We have shown, how geometrical algorithms can be used to solve the problem of finding operation points of a class of oscillating electronic circuits. Basically we show how to explicitly calculate "jump" sets on the state space Manifold that capture the behavior of the circuit. If it is possible to extend these methods to higher dimensional spaces, they can potentially be used as an alternative to SPICE-based circuit simulators in commercial software packages. Using the "Van der Pol"-example the solution procedure for an oscillator was explained.

To be successful, the following steps are necessary:
Based on classical theory of nonlinear electronical circuits, a differential-geometric basis has to developed. In contradiction to the currently used analytical approach, the differential-geometric basis allows the usage of numerical analysis methods
Furthermore representative benchmarks have to be constructed to validate the performance of the new mathematical methods. In particular, it will be possible to use algorithms to trace curves on surfaces in higher dimensions to find the operation points. Numerical differential-geometry methods for this case are already well provided and well understood.

The developement of efficient algorithms to calculate the solution curves on the surface being embedded in a n-dimensional space($n > 2$) is not easy, especially difficult will be to find sufficient starting points.

It will be essential to verify the calculated solutions for selected examples from real electronic circuits. Additionaly it will be very helpful to develop a system that is capable of visualizing the attributes of the circuits. In the context of complex nonlinear electornic circuits, their dynamic behaviour will be of great interest. New characteristics of the electronic circuits could be gained and used for design improvements.

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