Welfenlab
Institute of Man-Machine-Communication
Dr. Niklas Peinecke
Prof. Dr. Franz-Erich Wolter

Leibniz Universität Hannover

Eigenvalues of the Laplacian in computer vision

Can one hear the shape of an image?

A new image feature

- **Main goal**: Find a selected image within a large collection of probably similar images.
- **Problem**: Reduce the data thus reducing complexity of comparation without loss of accuracy of discrimination.
- Find a feature, that is both local **and** global in image data.

The spectrum of the Laplacian

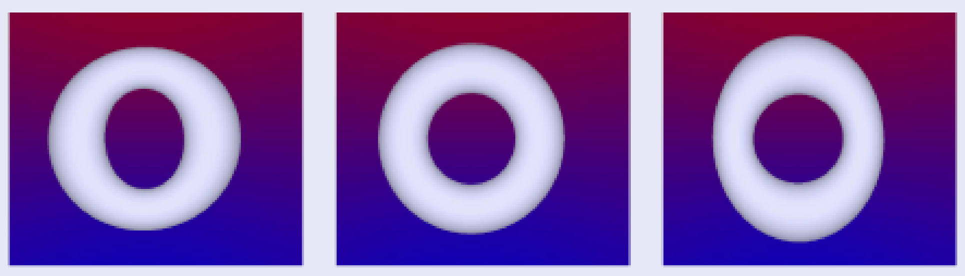
The **Laplacian** (also **Laplace-Beltrami operator**) on a manifold is defined by

$$\Delta f := \text{div grad } f$$

The **spectrum of the laplacian** is the set of eigenvalues of the Laplacian on a manifold, i.e. by solutions λ of the equation

$$\Delta f + \lambda f = 0$$

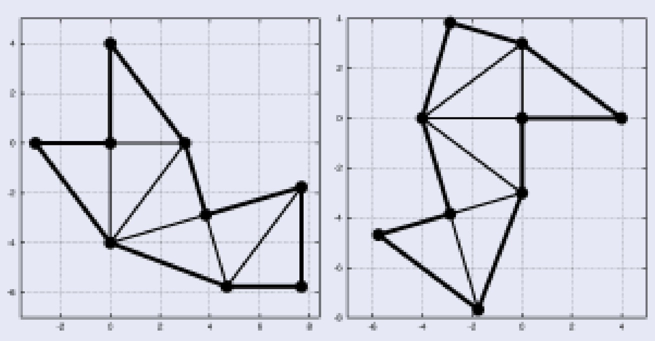
with Dirichlet boundary condition. Also known as **eigenfrequencies** because of their connection to oscillation problems of membranes $\rightsquigarrow$ **Can one hear the shape of a drum?** [L. Bers, cited in M. Kac, 1966]



An oscillating torus [Dennis Fehse, 2001]

Useful properties of the spectrum

- principally infinite, discrete:
  $0 \leq \lambda_1 < \lambda_2 < \ldots \uparrow +\infty$, each eigenspace is finite dimensional
- Isospectrality (i.e. two noncongruent shapes possessing identical spectra) is a rare phenomenon.

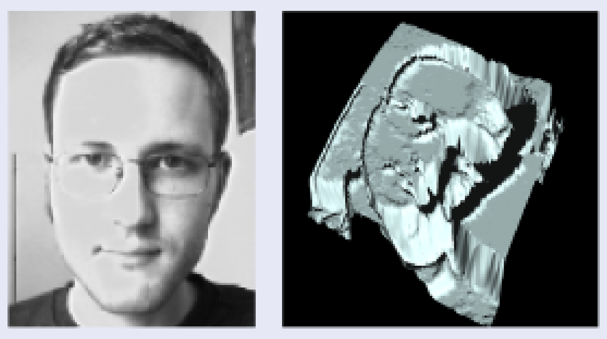


  The picture above shows an example of such shapes. Only few pairs of such shapes are known today.
- depending continously on the height function resp. the polygonal shape (see below)
- sensitive to local **and** global changes of the shape

Computing the spectrum

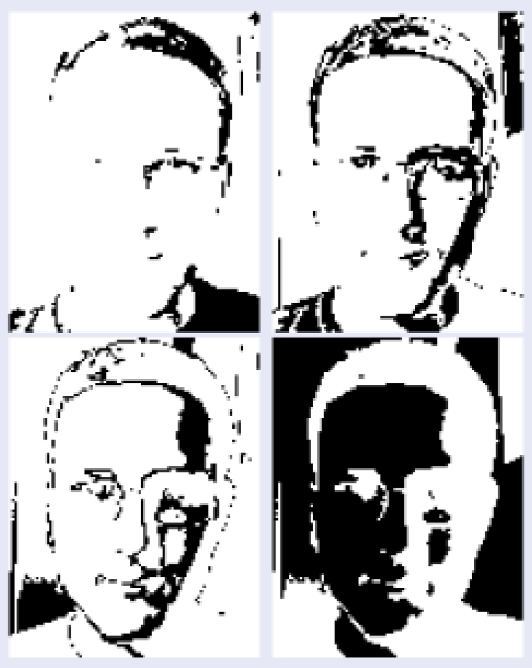
Two principal methods:

- Interpret a grey-value image as a height function, then solve the Laplace equation with FEM on an interpolating surface with linear, quadratic or cubic form functions.



Original image *Height function*

- Split the image into a sequence of polygonal shapes (isolines), solve the Laplace equation inside these shapes using singular elements as initiated by Descloux, Tolley [1983] and Driscoll [1997] (not yet implemented).



Sequence of polygonal layers

Color images

Color can be treated in two ways:

- Process each color channel separately; i.e. for RGB the spectrum becomes a sequence of 3D-vectors.
  Drawback: Influence of the (rather artificial, arbitraryly chosen) color-model
- Treat the image as a map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, i.e. a 2-manifold embedded in a 5D space, compute spectrum as in method 1.
  Drawback: Invariant to color rotations

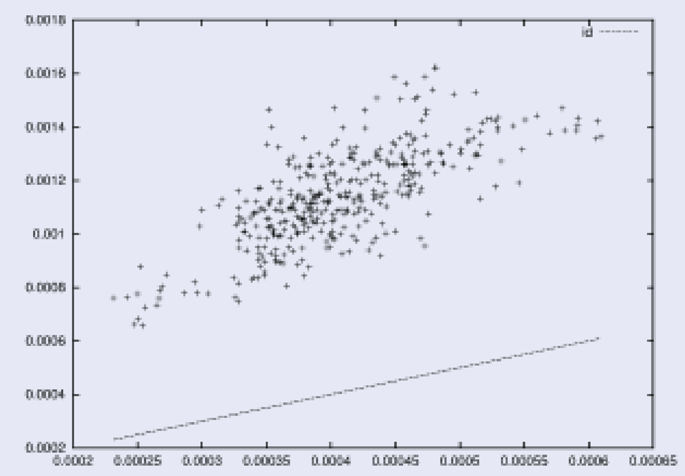
Color rotated images with identical spectra

Results

Method 1 gives good results in discrimination of a 600 images data set consisting of relatively similar facial images (Olivetti Research Laboratory test images):

typical grey value images

In this case the **first 4 eigenvalues** are sufficient for a relyable result.



Clustering of the first two eigenvalues in the Olivetti data base

Drawbacks:

- Slow for higher accuracy (thus precomputation necessary)
- Very large FEM matrices for "straight-forward-meshes" (typically $> 256^4$)
- Highly depending on the image size

Method 2 overcomes some of these drawbacks.

Outlook

- Preconditioning of eigenvalue problem / new (poss. orthogonal) formfunctions
- Improve computation (solve only for needed first few eigenvalues)
- Implement and test second method

References

- Peinecke, N.; Wolter, F.-E.; Reuter, M., Laplace-Spectra as Fingerprints for Image Recognition, Computer-Aided Design 6 (2007), no. 39, 460–476
- Peinecke, N.; Wolter, F.-E., Mass Density Laplace-Spectra for Image Recognition, in Proceedings of NASAGEM (2007), 409–416
- Wolter, F.-E.; Peinecke, N.; Reuter, M., *Verfahren zur Charakterisierung von Objekten / A Method for the Characterization of Objects (Surfaces, Solids and Images)*, German Patent Application, June 2005 (pending), US Patent US2009/0169050 A1, 2009

This work was supported by a grant of the DFG within the scope of the graduate college 615.